LOCAL FIELDS

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Apart from that absolute value $\|_\|\colon \mathbb{Q} \to \mathbb{R}_0^+$ it is possible to define a metric $\|_\|_p\colon \mathbb{Q} \to \mathbb{R}_0^+$ on the field of rational numbers for every prime number p making $(\mathbb{Q}, \|_\|_p)$ a metric field in which addition and multiplication are continuous. As a consequence its *completion* $(\mathbb{Q}_p, \|_\|_p)$ - the field of *p-adic numbers* - is a complete metric field which may be seen as an analogue of the real field $\mathbb{R}$, although it has also significantly different properties. E.g., it is locally compact, but totally disconnected; the unit ball

$$\mathbb{Z}_p = \{\, x \in \mathbb{Q}_p \mid \|x\|_p \leq 1 \,\}$$

is a subring of $\mathbb{Q}_p$ - the ring of *p-adic integers* - which is a *local ring* with maximal ideal $p\mathbb{Z}_p = \{\, x \in \mathbb{Q}_p \mid \|x\|_p < 1 \,\}$. It is the standard example of what is called a *non-archimedean local field*.

Local fields play an important role in many areas in mathematics. They are used in algebraic number theory in order to study the arithmetic of *number fields*, finite extensions of the field of rational numbers $\mathbb{Q}$. In the representation theory of finite groups they are used to link the *modular representation theory* of a finite group G with its characteristic 0 theory. They also appear in real analytic geometry on a very advanced level like in the description of the commensurability classes of *fundamental groups of certain locally symmetric spaces*.

In the course we will give a brief introduction to *local fields* including some analytical aspects like *Hensel's lemma* as well as some arithmetical aspects like *ramification* of finite extensions of local fields. In the final part of the course we will apply our acquired knowledge in order to construct counterparts of the so-called *symmetric spaces* for real semi-simple Lie groups: In a similar way as $\mathrm{GL}_2(\mathbb{C})$ acts on the Riemann sphere $\bar{\mathbb{C}}$ by *Möbius transformations*, $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on a locally-finite tree T which is the standard example of an *affine building*.

References

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