Calabi’s extremal Kähler metrics: An elementary introduction

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Introduction
CHAPTER 1

Elements of Kähler geometry

1.1. Kähler manifolds

An almost hermitian manifold $M = (M, g, J)$ is a manifold of (real) even dimension $n = 2m$, equipped with a riemannian metric $g$ and a $g$-compatible almost complex structure $J$. By definition, $J$ is a field of automorphisms of the tangent bundle $TM$ such that $J^2 = -I$, where $I$ denotes the identity — $TM$ can then be regarded as a complex vector bundle of rank $m$ — and $J$ is said $g$-compatible if the identity $g(JX, JY) = g(X, Y)$ holds for any two vector fields $X, Y$. The bilinear form $\omega$ defined by $\omega(X, Y) := g(JX, Y)$ is then a (skew-symmetric) 2-form and is called the Kähler form of the almost hermitian structure.

The Levi-Civita connection $D = D^g$ of a riemannian metric $g$ is the unique connection on $M$, acting on vector fields and naturally extended to any kind of tensor fields, which is torsion-free and preserves the metric, i.e. is such that $Dg = 0$. It can be expressed by the following Koszul formula:

$$2g(D_X Z, Y) = X \cdot g(Z, Y) + Z \cdot g(X, Y) - Y \cdot g(X, Z) + g([X, Z], Y) + g([Y, X], Z) + g(X, [Y, Z]),$$

for any vector fields $X, Y, Z$, where $[\cdot, \cdot]$ stands for the usual bracket of vector fields.

A Kähler manifold is then defined as an almost hermitian manifold $(M, g, J, \omega)$ such that $DJ = 0$. An equivalent formulation is given by the following proposition:

**Proposition 1.1.1.** An almost hermitian manifold $(M, g, J, \omega)$ is Kähler if and only if $J$ is integrable — so that $(M, J)$ is a complex manifold — and $\omega$ is closed — so that $(M, \omega)$ is a symplectic manifold.

**Proof.** By the celebrated Newlander-Nirenberg theorem [154], an almost complex structure $J$ is integrable if and only if the Nijenhuis tensor $N = N_J$ defined by

$$N(X, Y) = \frac{1}{4}([JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]),$$

is identically zero. The rhs can then be expressed in terms of the covariant derivative $DJ$ of $J$ and we then get

$$N(X, Y) = \frac{1}{4}((D_{JX} J)Y - J(D_X J)Y - (D_{JY} J)X + J(D_Y J)X).$$

We readily infer that $DJ = 0$ implies $N = 0$. By its very definition, the Kähler form $\omega$ is non-degenerate at each point of $M$. Moreover, its exterior derivative $d\omega$ is expressed in terms of its covariant derivative $D\omega$ by

$$d\omega(X, Y, Z) = (D_X \omega)(Y, Z) + (D_Y \omega)(Z, X) + (D_Z \omega)(X, Y),$$
Since the Levi-Civita connection $D$ preserves the metric $g$ the two conditions $DJ = 0$ and $D\omega = 0$ are equivalent and, by (1.1.4), imply that $\omega$ is closed, hence a symplectic 2-form. The converse is a consequence of the identity

$$g((DXJ)Y, Z) = \frac{1}{2}(d\omega(X, Y, Z) - d\omega(X, JY, JZ)) + 2g(JX, N(Y, Z)),$$

which can be easily checked by using (1.1.3)-(1.1.4). □

Remark 1.1.1. Formula (1.1.5) appears in [121], Vol. II; beware however that Kobayashi-Nomizu use different conventions for $N$ and for the exterior derivative; our own conventions will be made precise in the next sections.

Remark 1.1.2. An almost hermitian structure $(M, g, J, \omega)$ is called hermitian if $J$ is integrable, almost Kähler — or simply symplectic — if $\omega$ is closed (hence symplectic). By the above proposition, a Kähler structure can be alternatively defined as an almost hermitian structure which is both hermitian and symplectic. The following criterion is often useful:

Proposition 1.1.2. An almost hermitian structure $(M, g, J)$ is hermitian if and only if the following identity holds:

$$DJXJ = JDXJ,$$

for each vector field $X$.

Proof. By (1.1.3), (1.1.6) readily implies $N = 0$, whereas the converse follows from the reciprocal identity

$$\frac{1}{2}g((DXJ)Y, Z) + J(DJXJ)Y, Z) = g(JX, N(Y, Z)) - g(JY, N(Z, X)) - g(JZ, N(X, Y)),$$

whose easy verification is left to the reader. □

Another useful integrability criterion for $J$ is given by

Lemma 1.1.1. An almost complex structure $J$ is integrable if and only if the identity

$$LJXJ = JLXJ,$$

holds for any vector field $X$.

Proof. Here and henceforth $L_X$ denotes the Lie derivative along $X$, acting on any sort of tensor fields, cf. Section 1.4. In the current case we have that $(L_XJ)(Y) = [X, JY] - J[X, Y]$, so that

$$(LJXJ - JLXJ)(Y) = 4N(X, Y),$$

for any vector fields $X, Y$. The conclusion follows immediately. □

Until Chapter 9 and unless otherwise specified, $M = (M, g, J, \omega)$ will denote a connected Kähler manifold of (real) dimension $n = 2m$, with riemannian metric $g$, complex structure $J$ and Kähler (symplectic) form $\omega$. 
1.2. Riemannian and symplectic duality

In general, the *riemannian volume form*, $v_g$, of an oriented, $n$-dimensional riemannian manifold $(M, g)$ is determined by
\begin{equation}
  v_g(e_1, \ldots, e_n) = 1,
\end{equation}
for any (local) positively oriented orthonormal frame $\{e_1, \ldots, e_n\}$; with respect to any local, positively oriented, coordinate system $x_1, \ldots, x_n$, we then have
\begin{equation}
  v_g = |\partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_n| \, dx_1 \wedge \cdots dx_n,
\end{equation}
where $|\partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_n|$ denotes the norm with respect to $g$ of the $n$-multivector $\partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_n$, equal to the square root of the determinant of the matrix $g_{ij} := g(\partial/\partial x_i, \partial/\partial x_j)$, cf. Section 1.3.

In the current Kähler situation — more generally in the almost hermitian case — the riemannian volume form then coincides with the *symplectic volume form*, i.e.
\begin{equation}
  v_g = \omega^m/m!,
\end{equation}
(easy verification).

The riemannian duality is denoted by $\#$, in the direction $T^* M \rightarrow TM$, and by $\flat$ in the direction $TM \rightarrow T^* M$: for any 1-form $\alpha$ and any vector fields $X, Y$, we then have $\alpha(Y) = g(Y, \alpha^\flat)$ and $X^\flat(Y) = g(Y, X)$. The same symbols will be used for the induced isomorphisms from $\Lambda^p M = \Lambda^p(T^*M)$ to $\Lambda^p(TM)$ and vice versa.

The symplectic duality is similarly denoted by $\#_\omega$ from $T^* M$ to $TM$ and $\flat_\omega$ from $TM$ to $T^* M$, with $\alpha(Y) = \omega(Y, \alpha^\flat_\omega)$ and $X^{\flat_\omega}(Y) = \omega(Y, X)$. Equivalently,
\begin{equation}
  \alpha = -\iota_{\alpha^{\flat_\omega}} \omega, \quad X^{\flat_\omega} = -\iota_X \omega,
\end{equation}
for any 1-form $\alpha$ and any vector field $X$.

The action of $J$ on $T^* M$ is defined in such a way that the isomorphisms $\sharp_\omega$ and $\flat_\omega$, as well as $\sharp_\omega$ and $\flat_\omega$, are $J$-linear: for any 1-form we then set
\begin{equation}
  (J\alpha)(X) = \alpha(J^{-1} X) = -\alpha(JX).
\end{equation}
This operator is then extended to $p$-forms by
\begin{equation}
  (J\psi)(X_1, \ldots, X_p) = \psi(J^{-1} X_1, \ldots, J^{-1} X_p) = (-1)^p \psi(JX_1, \ldots, JX_p)
\end{equation}
($Jf := f$, if $f$ is a 0-form, i.e. a function).

As an operator acting on the space of $p$-forms, $J$ then satisfies $J^2 = (-1)^p$.

The riemannian and the symplectic duality are then related to each other by
\begin{equation}
  \psi^{\flat_\omega} = J \psi^\sharp,
\end{equation}
for each $p$-form $\psi$. In particular, the *symplectic gradient*, $\text{grad}_\omega f$, of a (real) function $f$, defined as the symplectic dual of $df$, is related to the (usual) riemannian gradient $\text{grad} f$ by
\begin{equation}
  \text{grad}_\omega f = J \text{grad} f.
The symplectic form $\omega$ allows one to define a Poisson bracket in the space $C^\infty(M, \mathbb{R})$ of smooth real functions on $M$. Recall that for any two real functions $f_1, f_2$ on any symplectic manifold $(M, \omega)$ of dimension $n = 2m$, the Poisson bracket $\{f_1, f_2\} = \{f_1, f_2\}_\omega$ is defined by

$$\{f_1, f_2\}_\omega = m df_1 \wedge df_2 \wedge \omega^{m-1}. \tag{1.2.9}$$

Equivalently,

$$\{f_1, f_2\} = \omega(\text{grad}_\omega f_1, \text{grad}_\omega f_2) = \text{grad}_\omega f_1 \cdot f_2 = - \text{grad}_\omega f_2 \cdot f_1. \tag{1.2.10}$$

Since $\omega$ is closed, we infer that

$$\text{grad}_\omega(\{f_1, f_2\}) = [\text{grad}_\omega f_1, \text{grad}_\omega f_2], \tag{1.2.11}$$

from which it readily follows that the Poisson bracket satisfies the Jacobi identity:

$$\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\} = 0. \tag{1.2.12}$$

The space $C^\infty(M, \mathbb{R})$, equipped with the Poisson bracket, is then a Lie algebra. The identity (1.2.11) then means that the linear map

$$f \mapsto \text{grad}_\omega f \tag{1.2.13}$$

is a Lie algebra morphism from the space $C^\infty(M, \mathbb{R})$, equipped with the Poisson bracket, to the space $\mathcal{X}(M)$ of vector fields, equipped with the usual bracket. Its range is the Lie algebra, $\mathfrak{ham}_\omega$, of hamiltonian vector fields on $(M, \omega)$, cf. Section 1.5.

### 1.3. Inner product and exterior derivative

The pointwise inner product of any kind of tensors is denoted by $(\cdot, \cdot)$. Except for $p$-forms, $(\cdot, \cdot)$ is the tensorial inner product, i.e. the inner product naturally induced by $g$ on the tensor algebra. For $p$-forms however we adopt the following different, but usual convention:

$$\alpha_1 \wedge \ldots \wedge \alpha_p, \beta_1 \wedge \ldots \wedge \beta_p = \det((\alpha_i, \beta_j)). \tag{1.3.1}$$

The same convention will be used for $p$-multivectors, i.e. for sections of $\Lambda^p(TM)$.

The inner product defined that way on the space of $p$-forms is equal to $\frac{1}{p!}$ times the tensorial inner product. In particular, for the Kähler form we have $|\omega|^2 = (\omega, \omega) = m$, while the tensorial square norm of $\omega$ is equal to $|g|^2 = 2m$. This choice is dictated by our convention for the exterior product, namely

$$(\alpha_1 \wedge \ldots \wedge \alpha_p)(X_1, \ldots, X_p) = \det((\alpha_i(X_j))), \tag{1.3.2}$$
and for the exterior derivative

\[
d\psi(X_0, X_1, \ldots, X_p) = \sum_{j=0}^{p} (-1)^j X_j \cdot \psi(X_0, \ldots, \hat{X}_j, \ldots, X_p) + \sum_{i<j} (-1)^{i+j} \psi([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p),
\]

where \( \hat{X}_i \) means that the corresponding argument has been omitted. These conventions have become widely used and there is no real need to change them (notice that these conventions amount to consider the exterior algebra bundle \( \Lambda^\bullet M \) as a quotient rather than a subbundle of the tensor algebra bundle).

For any local orthonormal frame \( \{e_1, \ldots, e_n\} \), we also have

\[
d\psi = \sum_{i=1}^{n} e_i^* \wedge D_{e_i} \psi,
\]

where \( \{e_1^*, \ldots, e_n^*\} \) denotes the (algebraic) dual coframe, determined by \( e_i^*(e_j) = \delta_{ij} \). Equivalently,

\[
d\psi(X_0, \ldots, X_p) = \sum_{j=0}^{p} (-1)^j D_{X_j} \psi(X_0, \ldots, \hat{X}_j, \ldots, X_p),
\]

i.e. \( d\psi \) is a normalized anti-symmetrization of the covariant derivative \( D\psi \) with respect to the Levi-Civita connection (or any torsionless connection).

For all kinds of tensor, the \textit{global} inner product is denoted by \( \langle \cdot, \cdot \rangle \):

\[
\langle \cdot, \cdot \rangle = \int_M \langle \cdot, \cdot \rangle v_g.
\]

When complex tensors are involved, unless otherwise specified \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle \) will denote the corresponding \textit{hermitian} inner products, as defined in Section 1.8.

1.4. Lie derivative and the Cartan formula

For any diffeomorphic map \( \Phi \) from a manifold \( M \) to a manifold \( N \), and for any tensor field \( T \) — of any sort — on \( M \), the \textit{direct image} \( \Phi \cdot T \) is the tensor field on \( N \) obtained by transporting the corresponding tensor fiber bundle over \( M \) to the tensor bundle of the same kind on \( N \) via the differential \( \Phi_* \) of \( \Phi \).

If \( X \) is a vector field on \( M \), \( \Phi \cdot X \) is then the vector field on \( N \) defined by \( (\Phi \cdot X)_y = \Phi_*(X_{\Phi^{-1}(y)}) \); if \( \psi \) is a \( p \)-form on \( M \), \( \Phi \cdot \psi \) is the \( p \)-form on \( N \) defined by \( (\Phi \cdot \psi)_y(Y_1, \ldots, Y_p) = \psi_{\Phi^{-1}(y)}(\Phi^{-1}_*Y_1, \ldots, \Phi^{-1}_*Y_p) \); if \( A \) is a field of endomorphisms of \( TM \), \( \Phi \cdot A \) is the field of endomorphisms of \( TN \) defined by \( (\Phi \cdot A)_y(Y) = \Phi_*(A_{\Phi^{-1}(y)}(\Phi^{-1}_*Y)) \), etc...

These definitions hold in particular when \( M = N \) and \( \Phi \) is a diffeomorphism of \( M \). We then get a group action of the group \( \text{Diff}(M) \) of diffeomorphisms of \( M \) on the space of tensor fields.
The **Lie derivative** of $T$ along a vector field $X$, denoted by $\mathcal{L}_X T$, is then defined by

$$\mathcal{L}_X T = -\frac{d}{dt}|_{t=0}(\Phi^X_t \cdot T),$$

where $\Phi^X_t$ denotes the (local) flow of $X$ in some neighbourhood of the point on consideration or, more generally, any curve $\Phi^X_t$ in $\text{Diff}(M)$ such that $\Phi^X_0 = I$, the identity, and $\frac{d}{dt}|_{t=0} \Phi^X_t = X$.

The sign in (1.4.1) has been chosen in such a way that $\mathcal{L}_X f = df(X)$ when $f$ is a function, and we then have

$$\mathcal{L}_X Y = [X, Y],$$

when $Y$ is a vector field.

By its very definition, the Lie derivative preserves each natural linear operation on tensors.

In particular $\mathcal{L}_X$ is a derivation of the exterior algebra in the sense that

$$\mathcal{L}_X (\psi \wedge \psi') = \mathcal{L}_X \psi \wedge \psi' + \psi \wedge \mathcal{L}_X \psi',$$

for any $p$-form $\psi$ and any $p'$-form $\psi'$, and commute with the exterior differential. These facts, together with the above identity $\mathcal{L}_X f = df(X)$ imply the following Cartan formula:

$$\mathcal{L}_X \psi = \iota_X d\psi + d(\iota_X \psi),$$

for any $p$-form $\psi$ (both sides are derivations of the exterior algebra, commute to $d$ and are equal to $df(X)$ when $\psi = f$ is a function; they then have the same action on the whole exterior algebra).

In (1.4.4) and henceforth, $\iota_X \psi$ denotes the **interior product** of $\psi$ by $X$, i.e., if $p > 0$, the $(p - 1)$-form defined by

$$\iota_X \psi(X_1, \ldots, X_{p-1}) = \psi(X, X_1, \ldots, X_{p-1}),$$

for any vector fields $X, X_1, \ldots, X_{p-1}$ (we agree that $\iota_X f = 0$ if $f$ is a function).

The Lie derivative is a Lie algebra action: for any two vector fields $X, Y$, and for any kind of vector field $T$, we have that

$$\mathcal{L}_{[X, Y]} T = \mathcal{L}_X (\mathcal{L}_Y T) - \mathcal{L}_Y (\mathcal{L}_X T).$$

### 1.5. Symplectic vector fields and Poisson brackets

If $(M, \omega)$ is a symplectic manifold, a vector field $X$ is called **symplectic** if

$$\mathcal{L}_X \omega = 0.$$

By the Cartan formula (1.4.4), this condition amounts to $d(\iota_X \omega) = 0$. A vector field $X$ is then symplectic if and only if its symplectic dual $X^\flat \omega$ is closed. It is called **hamiltonian** if $X^\flat \omega$ if exact. We then have

$$X = \text{grad}_\omega h^X,$$

i.e.

$$\iota_X \omega = -dh^X,$$

if $(M, \omega)$ is a symplectic manifold, a vector field $X$ is called **symplectic** if

$$\mathcal{L}_X \omega = 0.$$
where $h^X$ is a real function and $\text{grad}_\omega X$ is the symplectic gradient of $X$ defined in Section 1.2: $h^X$ is called the \textit{momentum} of $X$ with respect to $\omega$ and is uniquely defined by $X$ up to an additive constant.

By (1.4.6), the space of symplectic vector fields is a Lie algebra, denoted $\mathfrak{sp}_\omega$. By (1.2.11) the space of hamiltonian vector fields is a Lie algebra of $\mathfrak{sp}_\omega$, isomorphic to the space $C^\infty(M, \mathbb{R})/\mathbb{R}$ equipped with the Poisson bracket defined in Section 1.2; it will be denoted by $\mathfrak{ham}_\omega$.

**Proposition 1.5.1.** The Lie algebra $\mathfrak{ham}_\omega$ is an ideal of $\mathfrak{sp}_\omega$ and contains its derived ideal:

\begin{equation}
[\mathfrak{sp}_\omega, \mathfrak{sp}_\omega] \subset \mathfrak{ham}_\omega.
\end{equation}

More precisely, for any two elements $X, Y$ of $\mathfrak{sp}_\omega$, we have:

\begin{equation}
[X, Y] = \text{grad}_\omega (\omega(X,Y)).
\end{equation}

**Proof.** It suffices to check (1.5.5). By the Cartan formula, for any vector fields $X, Y$ we have

\begin{equation}
d(\omega(X,Y)) = -d(\iota_X \iota_Y \omega)
= -\mathcal{L}_X (\iota_Y \omega) + \iota_X (d(\iota_Y \omega))
= -\iota_{[X,Y]}\omega - \iota_Y \mathcal{L}_X \omega + \iota_X \mathcal{L}_Y \omega.
\end{equation}

When $X, Y$ both belong to $\mathfrak{sp}_\omega$, this reduces to

\begin{equation}
\iota_{[X,Y]}\omega = -d(\omega(X,Y)),
\end{equation}

which is the same as (1.5.5). \hfill \Box

In the case that $X = \text{grad}_\omega h^X$ and $Y = \text{grad}_\omega h^Y$ both belong to $\mathfrak{ham}_\omega$, then $\omega(X,Y) = \{h^X, h^Y\}$ and (1.5.5) then reduces to (1.2.11).

### 1.6. Cauchy-Riemann operators and Chern connections

The material presented in this section and in the next Section 1.18 is of quite general character and, unless specifically stated, does not require $M$ being Kähler.

We start this section by recalling some general facts concerning the general theory of linear connections on vector bundles.

Let $E$ be any real or complex vector bundle over some $n$-dimensional manifold $M$. A \textit{linear connection} on $E$ or, rather, the corresponding \textit{covariant derivative}, is a first-order linear differential operator $\nabla$ acting on sections of $E$, with values in the tensor product $T^* M \otimes E$, which satisfies the following \textit{Leibniz identity}:

\begin{equation}
\nabla(fs) = df \otimes s + f \nabla s,
\end{equation}

for any section $s$ of $E$ and any (real or complex) function $f$ (when $E$ is a complex vector bundle, $T^* M \otimes E$ is implicitly understood as the tensor product over $\mathbb{C}$ of the complexified cotangent bundle $T^* M \otimes \mathbb{C}$ with $E$).

Any linear connection $\nabla$ extends into an “exterior differential” $d^\nabla$ acting on $E$-valued \textit{exterior forms}, i.e. on sections of the tensor product $E \otimes \Lambda^\bullet M$, where $\Lambda^\bullet M = \sum_{p=0}^n \Lambda^p M$, denotes the bundle of (real) exterior forms on $M$ (again, if $E$ is a complex vector bundle, $E \otimes \Lambda^\bullet M$ is tacitly understood as
the tensor over \( \mathbb{C} \) of \( E \) with the vector bundle \( \Lambda^* M \otimes \mathbb{C} \) of complex exterior forms). This is done by setting

\[
(1.6.2) \quad d^\nabla (s \otimes \phi) = \nabla s \wedge + s \otimes d\phi,
\]

for any decomposed section \( s \otimes \phi \) of \( E \otimes \Lambda^* M \). Equivalently, for any \( E \)-valued \( p \)-form \( \phi \), \( d^\nabla \phi \) is defined by

\[
(1.6.3) \quad (d^\nabla \phi)(X_0, X_1, \ldots, X_p) = \sum_{i=0}^{p} (-1)^i \nabla_{X_i} \phi(X_0, \ldots, \hat{X}_i, \ldots, X_p))
+ \sum_{i<j} (-1)^{i+j} \phi([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p),
\]

for any vector fields \( X_0, X_1, \ldots, X_p \) (compare with (1.3.3)). For any auxiliary torsion-free linear connection \( D \) on \( M \), denote by \( D^\nabla \) the induced covariant derivative acting on the space of \( E \)-valued exterior forms by \( D^\nabla (s \otimes \phi) = \nabla s \otimes \phi + s \otimes D\phi \); then, the above formula can be rewritten as

\[
(1.6.4) \quad (d^\nabla \phi)_{X_0, X_1, \ldots, X_p} = \sum_{k=0}^{p} (-1)^k (D^\nabla_{X_k} \phi)_{X_0, \ldots, \hat{X}_k, \ldots, X_p},
\]

or, in a more concise way:

\[
(1.6.5) \quad d^\nabla \phi = \sum_{i=1}^{n} e^*_i \wedge D^\nabla_{e_i} \phi,
\]

for any auxiliary frame \( \{e_1, \ldots, e_n\} \) of \( TM \). Denote by \( \text{End}(E) \) the vector bundle of \( \mathbb{R} \)- or \( \mathbb{C} \)-linear endomorphisms of \( E \), according to \( E \) being a real or a complex vector bundle. Then, the difference \( \nabla' - \nabla \) of any two linear connections \( \nabla, \nabla' \) is a \( \text{End}(E) \)-valued 1-form, whereas the curvature, \( R^\nabla \), of \( \nabla \) is a \( \text{End}(E) \)-valued 2-form, determined by

\[
(1.6.6) \quad R^\nabla_{X,Y} s = (\nabla_{[X,Y]} - [\nabla_X, \nabla_Y]) s,
\]

for any vector fields \( X, Y \) on \( M \) and any section \( s \) of \( E \). The curvature \( R^\nabla \) then appears as the obstruction for \( d^\nabla \) to be a cohomological operator as the usual exterior differential, via the following — easily checked — identity:

\[
(1.6.7) \quad (d^\nabla \circ d^\nabla) \phi = -R^\nabla \wedge \phi,
\]

for any \( E \)-valued \( p \)-form \( \phi \) (in the rhs of (1.6.7) \( R^\nabla \wedge \phi \) stands for the \( E \)-valued \((p+2)\)-form obtained by performing the formal exterior product of \( R^\nabla \) and \( \phi \), with values in the tensor product \( \text{End}(E) \otimes E \), followed by the evaluation map \( \text{End}(E) \otimes E \to E \)).

A linear connection \( \nabla \) is said to be integrable — or flat — if \( R^\nabla \equiv 0 \), equivalently if \( d^\nabla \circ d^\nabla = 0 \).

To any linear connection \( \nabla \) on \( E \) is attached a horizontal distribution, \( H^\nabla \), defined on the total space of \( E \) as follows: for each \( \xi \) in \( E \) over \( x \), \( H^\nabla_\xi \) is defined by \( H^\nabla_\xi = s_*(T_x M) \subset T_\xi E \) for any local section \( s \) of \( E \) such that \( \nabla s = 0 \) at \( x \), where \( s \) is viewed as a map from \( M \) to \( E \) and \( s_\ast \) stands for its
differential; in other words, the pair \((\xi, H^\nabla_\xi)\) is the 1-jet at \(x\) of any section of \(E\) which is \(\nabla\)-parallel at \(x\). The tangent space \(TE\) then splits as

\[ TE = H^\nabla \oplus T^V E, \]

where \(T^V\) stands for the vertical tangent space, whose fiber at \(\xi\) is \(T^V_\xi E := T_\xi(E_x) \cong E_x\). For any section \(s\) of \(E\), again viewed as a map from \(M\) to \(E\), and for any vector \(X\) in \(T_xM\), the value of the covariant derivative \(\nabla_X s\) at \(x\) is then given by

\[ (\nabla_X s)(x) = v^\nabla(s_*(X)), \]

where \(v^\nabla : TE \to T^V E\) denotes the projection from \(TE\) to \(T^V E\) along \(H^\nabla\); equivalently,

\[ s_*(X) = \tilde{X} + \nabla_X s, \]

for any vector field \(X\) on \(M\), if \(\tilde{X}\) denotes the horizontal lift of \(X\) on \(E\) with respect to \(H^\nabla\), i.e. the unique section of \(H^\nabla\) which projects to \(X\). Alternatively, any section \(s\) of \(E\) can be regarded as a vertical vector field on the total space of \(E\), constant on each fibre, and we then have

\[ \nabla_X s = [\tilde{X}, s]. \]

By using (1.6.10), one easily checks that the curvature \(R^V\) can then be alternatively defined by

\[ R^V_{X,Y}\xi = v^\nabla([\tilde{X}, \tilde{Y}]_\xi), \]

for any two vectors \(X,Y\) in \(T_xM\) and any \(\xi \in E_x\) (in the rhs of (1.6.12), \(\tilde{X}, \tilde{Y}\) are the horizontal lifts of any extensions of \(X,Y\) and \([\tilde{X}, \tilde{Y}]_\xi\) denotes the value at \(\xi\) of their bracket on \(E\)). The vanishing of \(R^V\) is thus equivalent to the integrability of \(H^\nabla\) in the sense of the Frobenius theorem. It follows that a linear connection \(\nabla\) on \(E\) is integrable if and only if \(E\) can be locally trivialized by \(\nabla\)-parallel sections (for a more detailed discussion we refer the reader to [171] or to our notes [90]).

**Remark 1.6.1.** The sign convention adopted in (1.6.6), hence also in (1.6.7) and (1.6.12), for the curvature of a linear connection suits with [28] and will be followed throughout these notes. The reader is warned however that the opposite convention: \(R^V_{X,Y} s = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]} ) s\) is commonly encountered in the more recent literature.

We now restrict our attention to the case when \(E\) is a complex vector bundle of (complex) rank \(r\) defined over an almost complex manifold \((M,J)\).

A Cauchy-Riemann operator \(\bar{\partial}^E\) on \(E\) is then defined as a first order \(\mathbb{C}\)-linear differential operator acting on sections of \(E\) with values in the (complex) tensor product \(E \otimes \Lambda^{0,1}M\), satisfying the following Leibniz-like identity

\[ \bar{\partial}(fs) = s \otimes \bar{\partial} f + f \bar{\partial}^E s, \]

for any section \(s\) of \(E\) and any (complex-valued) function \(f\) on \(M\). Here, \(\bar{\partial} = \frac{1}{2}(d - id^c)\) denotes the usual Cauchy-Riemann operator acting on functions (note however that in the general almost complex context, we don’t have \(\bar{\partial} \circ \bar{\partial} = 0\), as \(d, d^c\) don’t commute, cf. Section 1.11).
Any $\mathbb{C}$-linear connection $\nabla$ on $E$ can be written as
\begin{equation}
\nabla = \nabla^{1,0} + \nabla^{0,1},
\end{equation}
where the $(1, 0)$-part $\nabla^{1,0}$ and the $(0, 1)$-part $\nabla^{0,1}$ are defined by
\begin{equation}
\nabla^{1,0}_X s = \frac{1}{2}(\nabla_X s - i\nabla_{JX}s), \quad \nabla^{0,1}_X s = \frac{1}{2}(\nabla_X s + i\nabla_{JX}s),
\end{equation}
and $\nabla^{0,1}$ clearly satisfies (1.6.13), hence is a Cauchy-Riemann operator. Conversely, any Cauchy-Riemann operator can be realized as the $(0, 1)$-part of a $\mathbb{C}$-linear connection in several ways. If $E$ is equipped with a hermitian inner product, $h$, a $\mathbb{C}$-linear connection $\nabla$ is called hermitian if it preserves $h$, i.e., if
\begin{equation}
X \cdot h(s_1, s_2) = h(\nabla_X s_1, s_2) + h(s_1, \nabla_X s_2),
\end{equation}
for any sections $s_1, s_2$ of $E$ and any (real) vector field $X$. We then have

**Proposition 1.6.1.** Let $E$ be a complex vector bundle over an almost complex manifold $(M, J)$. Then, for any given hermitian inner product $h$ on $E$, any Cauchy-Riemann operator $\bar{\partial}^E$ on $E$ is the $(0, 1)$-part of a unique hermitian connection $\nabla$.

**Proof.** We first prove uniqueness. If two $\mathbb{C}$-linear connections have the same $(0, 1)$-part, their difference, say $\eta$, is of type $(1, 0)$, as a $\text{End} E$-valued $1$-form, hence satisfies the identity $\eta_{JX} = i\eta_X$; if, moreover, both connections preserve $h$, then $\eta_X$ is $h$-skew hermitian for any vector $X$: these two properties are clearly incompatible, unless $\eta$ is identically zero.

To construct the connection, we first observe that any Cauchy-Riemann operator $\bar{\partial}^E$ defined on $E$ induces a Cauchy-Riemann operator $\bar{\partial}^{E^*}$ defined on the (complex) dual $E^*$ by the following duality formula:
\begin{equation}
\langle \bar{\partial}^{E^*} \sigma, s \rangle = \bar{\partial} \langle \sigma, s \rangle - \langle \sigma, \bar{\partial}^E s \rangle,
\end{equation}
for any section $\sigma$ of $E^*$ and any section $s$ of $E$, where $\langle \cdot, \cdot \rangle$ stands for the duality between $E$ and $E^*$. Denote by $\tau$ the hermitian duality from $E$ to $E^*$ defined by $\langle \xi_1, \tau(\xi_2) \rangle = h(\xi_1, \xi_2)$. We then define a connection $\nabla$ by
\begin{equation}
\nabla s = \tau^{-1} (\bar{\partial}^{E^*} \tau(s)) + \bar{\partial}^E s,
\end{equation}
for any section $s$ of $E$. By its very definition, $\nabla$ is $\mathbb{C}$-linear and preserves the hermitian inner product $h$, hence is a hermitian connection on $E$, whereas we clearly have $\nabla^{0,1} = \bar{\partial}^E$. 

When $(M, J)$ is a complex manifold and $E$ is a holomorphic vector bundle over $M$, then $E$ admits an (integrable) canonical Cauchy-Riemann operator $\bar{\partial}^E$, defined below, and for any hermitian inner product $h$, the resulting hermitian connection defined by Proposition 1.6.1 is usually called the Chern connection of the hermitian holomorphic vector bundle $(E, h)$.

In the more general context of this section, $\nabla$ will still be called the Chern connection of the hermitian vector bundle $(E, h)$ with respect to the chosen Cauchy-Riemann operator $\bar{\partial}^E$.

In the sequel of this section, we assume that $J$ is integrable.

Then, any Cauchy-Riemann operator $\bar{\partial}^E$ extends to an operator, still denoted by $\bar{\partial}^E$, acting on sections of $E \otimes \Lambda^{0,*} M$, where $\Lambda^{0,*} M$ denotes the
1.6. CAUCHY-RIEMANN OPERATORS AND CHERN CONNECTIONS

bundle of (complex-valued) forms of type \((0, q)\) on \(M\), cf. Section 1.9 below. The action of \(\bar{\partial}^E\) on sections of \(E \otimes \Lambda^{0,*}M\) is determined by its action on decomposed (local) section \(s \otimes \phi\) of \(E \otimes \Lambda^{0,*}M\), defined by:

\[
\bar{\partial}^E(s \otimes \phi) = \bar{\partial}E s \wedge \phi + s \otimes \bar{\partial}\phi
\]  

(compare with (1.6.2)).

A Cauchy-Riemann operator \(\bar{\partial}^E\) is said to be integrable if its “curvature” vanishes, i.e.

\[
\bar{\partial}^E \circ \bar{\partial}^E = 0.
\]

A holomorphic vector bundle is defined as a vector bundle \(E\) in the category of holomorphic manifolds. This means that \(E\) is a complex vector bundle over some complex manifold \((M, J)\), which, as a manifold, is itself equipped with a complex structure in such a way that all fibers are complex submanifolds of \(E\) and the two operations which make \(E\) into a complex vector bundle, namely the addition and the product by a complex number, are both holomorphic; moreover, \(E\) can be locally trivialized in the usual sense by (local) holomorphic sections, i.e. by local sections which are holomorphic as (local) maps from \(M\) to \(E\).

Any holomorphic (complex) vector bundle \(E\) over a complex manifold \((M, J)\) admits a canonical Cauchy-Riemann operator \(\bar{\partial}^E\) defined by

\[
\bar{\partial}^E s = \sum_{i=1}^r \bar{\partial} f_i \otimes s_i,
\]

for any section \(s\) of \(E\), locally written as \(s = \sum_{i=1}^m f_i s_i\) with respect to a holomorphic local frame \(s_i\) of \(E\), where \(\bar{\partial} = \frac{1}{2} (d - id^c)\) denotes the usual Cauchy-Riemann operator acting on functions. The Cauchy-Riemann operator \(\bar{\partial}^E\) defined that way is independent of the chosen holomorphic frame \(s_i\) and is evidently integrable, as \(\bar{\partial} \circ \bar{\partial} = 0\). It is characterized by the fact that a (local) section of \(E\) is holomorphic — as a map from an open set of \(M\) to \(E\) — if and only if it belongs to the kernel of \(\bar{\partial}^E\). The converse is the celebrated Koszul-Malgrange theorem:

\[
\text{Theorem 1.6.1 (Koszul-Malgrange [124])}. \text{ Let } (M, J) \text{ be a complex manifold of real dimension } n = 2m \text{ and } E \text{ a complex vector bundle over } M. \text{ Let } \bar{\partial}^E \text{ be a Cauchy-Riemann operator on } E \text{ satisfying the integrability condition } (1.6.20). \text{ Then, } E \text{ admits a unique structure of holomorphic vector bundle whose canonical Cauchy-Riemann operator is } \bar{\partial}^E.
\]

\[
\text{Proof.} \text{ The proof relies on the above construction, which makes Theorem 1.6.1 a direct consequence of the Newlander-Nirenberg theorem along the following line of argument. We first chose an auxiliary hermitian inner product on } E \text{ and we complete } \bar{\partial}^E \text{ into a hermitian connection } \nabla \text{ according to the above construction. The tangent space } TE \text{ then splits as in (1.6.8) and we construct an almost-complex structure, say } J^E, \text{ en } E \text{ by transporting the complex structure } J \text{ of } M \text{ on the horizontal, via the isomorphisms } H^\nabla_{\xi} = T_\xi M, \text{ and the complex structure of the fibres of } V \text{ on } V. \text{ We then check that } J^E \text{ is independent of the chosen auxiliary hermitian inner product and that the Nijenhuis tensor of } J^E \text{ is essentially identified with the square}
\]
\[ \partial E \circ \bar{\partial} E \text{ which, up to sign, is the } (0,2)\text{-part of the curvature of } \nabla: \text{ by the Newlander-Nirenberg theorem, } J^E \text{ is then induced by a complex structure of } E. \text{ It is then easy to show that this complex structure actually makes } E \text{ into a holomorphic vector bundle over } M. \]

In view of Theorem 1.6.1, an integrable Cauchy-Riemann structure on a complex vector bundle \( E \) over a complex manifold \((M, J)\) is called a holomorphic structure on \( E \). We then have

**Proposition 1.6.2.** Let \( E \) be a complex vector bundle over a complex manifold \((M, J)\). For any fixed hermitian inner product \( h \) on \( E \), the map \( \nabla \mapsto \nabla^0,1 \) establishes a natural one to one correspondence between the set of all holomorphic structures of \( E \) and the set of all hermitian connections whose curvature is \( J \)-invariant.

**Proof.** For any \( C \)-linear connection \( \nabla \), it follows from (1.6.7) that its curvature can be expressed as \( R^\nabla = -d \circ d \nabla \), whose \((0,2)\)-part is \( -d \circ d \nabla^{0,1} \). It follow that the Cauchy-Riemann operator \( \nabla^{0,1} \) is integrable if and only if the \((0,2)\)-part of \( R^\nabla \) is zero. If \( \nabla \) is hermitian, the \((2,0)\)- and \((0,2)\)-parts of \( R^\nabla \) are related to each other by the identity \( R^\nabla_{Z_1, Z_2} = -R^\nabla_{Z_2, Z_1}^* \), for any two complex vector fields \( Z_1, Z_2 \), where \( * \) stands for the hermitian adjoint: then, \( \nabla^{0,1} \) is integrable if and only if \( R^\nabla \) is reduced to its \( J \)-invariant part, i.e. satisfies \( R^\nabla_{JX, JY} = R^\nabla_{XY} \). We then conclude by using Proposition 1.6.1. \[\Box\]

### 1.7. Chern connections of complex line bundles

In this section, we apply the theory of Chern connections developed in Section 1.6 to the particular case when \( E \) is a complex line bundle, i.e. a complex vector bundle of rank 1. As such, it will be generally denoted by \( L \), instead of \( E \), and the Chern connection of a hermitian holomorphic line bundle will be generally denoted by \( \nabla \). The curvature, \( R^\nabla \), of \( \nabla \) is then of the form \( R^\nabla = i \rho^\nabla \), where \( \rho^\nabla \) is a real \( 2 \)-form, called the curvature form of \( \nabla \), or of \((L, h)\).

Denote by \( \tilde{\pi} \) the holomorphic fiber bundle obtained by removing the zero section, \( \Sigma_0 \), of \( L \): \( \tilde{\pi} \) can then be viewed as the associated \( (\text{holomorphic}) \ C^*\)-principal bundle; the restriction of \( \pi \) to \( \tilde{\pi} \) is denoted \( \tilde{\pi} \).

**Proposition 1.7.1.** Let \((L, h)\) be a holomorphic line bundle over a complex manifold \((M, J)\). Denote by \( s \) any non-vanishing holomorphic section of \( L \), i.e. any local holomorphic section of \( \tilde{\pi} \) defined on some open subset \( U \) of \( M \). Then, on \( U \), the Chern connection \( \nabla \) and the curvature form \( \rho^\nabla \) have the following expressions:

\[
\nabla s = \partial \log |s|^2_h \otimes s = \frac{1}{2} \left( d \log |s|^2_h + i d^c \log |s|^2_h \right) \otimes s
\]

where \( |s|^2_h \) stands for \( h(s, s) \), and

\[
\rho^\nabla = -\frac{1}{2} dd^c \log |s|^2_h.
\]

Moreover, we have that

\[
\tilde{\pi}^* \rho^\nabla = -dd^c \log r,
\]
where \( r \) stands the norm function on \( L \) determined by \( h \), here restricted to \( \tilde{L} \).

**Proof.** For simplicity, we denote by \((\cdot, \cdot)\) the hermitian inner product \( h \). Since \( \nabla \) preserves \( h \), we have that \((\nabla_X s, s) = X \cdot |s|^2_h - (s, \nabla_X s)\) and, similarly, \((\nabla_{JX} s, s) = JX \cdot |s|^2_h - (s, \nabla_{JX} s)\) for any (real) vector field \( X \). Since \( s \) is a holomorphic section and \( \nabla^{0,1} \) is equal to the Cauchy-Riemann operator which determines the holomorphic structure of \( L \), we have that \( \nabla_{JX} s = i \nabla_X s \), so that the latter identity can be rewritten as \((\nabla_{JX} s, s) = JX \cdot |s|^2_h + i (s, \nabla_X s) \). Now, \( L \) is a complex line bundle: we then have that \( \nabla_X s = \theta(X) s \), for some complex 1-form \( \theta \) defined on \( \mathcal{U} \). By combining the above two identities, we immediately infer that \( \theta = \frac{1}{2} (d \log |s|^2_h + i \text{d} \text{d} \log |s|^2_h) = \partial X \log |s|^2_h \). This proves (1.7.1). By the very definition of \( R^\nabla \) — see Remark 1.6.1 below — we have that \( R^\nabla s = -d \theta \otimes s \equiv -\frac{1}{2} \text{d} \text{d} \log |s|^2_h \otimes i s \); this proves (1.7.2). If \( s \) is regarded as a (holomorphic) map from \( \mathcal{U} \) to \( \tilde{L} \), (1.7.2) can alternatively be read as: \( \rho^\nabla = -i \text{d} \text{d} \log r \equiv s^* (-i \text{d} \text{d} \log r) \). Now, on \( \tilde{L} \), the 2-form \(-i \text{d} \text{d} \log r \) is basic, i.e. \(-i \text{d} \text{d} \log r = \tilde{\pi}^* \psi \), for some 2-form \( \psi \) defined on \( M \); this is because, if \( T \) denotes the generator of the natural \( S^1 \)-action on \( \tilde{L} \), we have that \( d \text{d} \log r(T) = 1 \), \( d \log r(T) = 0 \), so that \( \iota_T i \text{d} \text{d} \log r = \iota_T \iota_T i \text{d} \text{d} \log r = 0 \), and \( \mathcal{L}_T i \text{d} \text{d} \log r = \mathcal{L}_T \iota_T i \text{d} \text{d} \log r = 0 \). Since \( \tilde{\pi} \circ s = \text{Id}_{\mathcal{U}}, \) we infer \( \psi^\nabla = s^* (-i \text{d} \text{d} \log r) = s^* \tilde{\pi}^* \psi = \psi \), on \( \mathcal{U} \), hence on \( M \), as \( \rho^\nabla \) and \( \psi \) are both globally defined on \( M \), and \( \mathcal{U} \) can be chosen a neighbourhood of any point of \( M \); we thus obtain (1.7.3). \( \square \)

As a direct corollary, we get

**Proposition 1.7.2.** Let \( L \) be a holomorphic line bundle over a complex manifold \((M, J)\). For any two hermitian inner product \( h, h' = e^{-2\phi} h \) on \( L \), the corresponding Chern connections \( \nabla \), \( \nabla' \) and the corresponding curvature forms \( \rho^\nabla \), \( \rho^\nabla' \) are related by

\[
\nabla' = \nabla - d\phi - i\text{d}\text{d}\phi, \tag{1.7.4}
\]

and

\[
\rho^\nabla' = \rho^\nabla + i\text{d}\text{d}\phi. \tag{1.7.5}
\]

**Proof.** The difference \( \nabla' - \nabla \) is naturally viewed as a complex 1-form. By (1.7.1) this complex 1-form is easily identified with the rhs of (1.7.4), as \( \log |s|^2_h = \log |s|^2_{h'} - 2\phi \); (1.7.5) then follows from (1.7.4) (or from (1.7.3)). \( \square \)

**Remark 1.7.1.** It follows from (1.7.2) and from (1.7.4) that for any holomorphic line bundle \( L \) and any hermitian inner product \( h \) on \( L \), the curvature form \( \rho^\nabla \) of the resulting Chern connection \( \nabla \) is closed and that for any two hermitian inner products on \( L \) the resulting Chern curvature induces the same element in the de Rham cohomology space \( H^2_{dR}(M, \mathbb{R}) \). This actually has nothing to do with the holomorphic structure of \( L \), only with its structure of complex line bundle: for any \( \mathbb{C} \)-linear connection \( \nabla \) on \( L \), the curvature \( R^\nabla \) is a complex 2-form and it is easily checked that \( -i R^\nabla \) determines the same element as the former ones in \( H^2_{dR}(M, \mathbb{R}) \). The resulting element in \( H^2(M, \mathbb{R}) \) is equal to \( 2\pi c_1(L) \), where \( c_1(L) \) is the real Chern class of \( L \). The factor \( 2\pi \) is there to ensure that \( c_1(L) \) be the image
1.8. Real vs complex viewpoint

The (real) tangent bundle \( TM \) of any almost-complex manifold \( (M, J) \) can be regarded as a rank \( m \) complex vector bundle by identifying the action of \( J \) with the multiplication by the complex number \( i \). As such, \( TM \) will be denoted by \( (TM, J) \).

The complex vector bundle \( (TM, J) \) is isomorphic to the bundle \( T^{1,0}M \) of complex vectors of type \((1,0)\) defined as follows: the operator \( J \) extends by \( \mathbb{C} \)-linearity to the complexified tangent bundle \( TM \otimes \mathbb{C} \) and \( T^{1,0}M \), resp. \( T^{0,1}M \), are the complex eigensubbundles of \( TM \otimes \mathbb{C} \) for the extended operator \( J \), corresponding to the eigenvalue \( i \), resp. \( -i \). Then, \( (TM, J) \) is identified with \( T^{1,0}M \) via the map: \( X \to X^{1,0} := \frac{1}{2}(X - iJX) \).

If the Nijenhuis tensor of \( J \) is identically zero, then \( J \) is integrable — cf. Section 1.1 — meaning that \( M \) is a complex manifold, whose structure is determined by local holomorphic complex coordinates which are deduced from each other by holomorphic transformations. Let \( z_1, \ldots, z_m \) be such a system of holomorphic coordinates on some open set \( U \) of \( M \) and, for \( j = 1, \ldots, m \), write \( z_j = x_j + iy_j \), where \( x_j, y_j \) are real. Then, \( x_1, y_1, \ldots, x_m, y_m \) form a system of real coordinates on \( U \). Denote by \( \partial/\partial x_j, \partial/\partial y_j \) the corresponding vector fields: as operators acting on functions, these are the partial derivatives with respect to the coordinate system \( x_j, y_j \). Since the \( z_j \) are holomorphic, we have that \( \partial/\partial y_j = J\partial/\partial x_j \) for all \( j = 1, \ldots, m \). We then define \( \partial/\partial z_j := \frac{1}{2}(\partial/\partial x_j - i\partial/\partial y_j) \) as the \((1,0)\)-part of \( \partial/\partial x_j \) and \( \partial/\partial \bar{z}_j := \frac{1}{2}(\partial/\partial x_j + i\partial/\partial y_j) \) as the \((0,1)\)-part of \( \partial/\partial x_j \).

Remark 1.8.1. Despite the notation, the local complex vector fields \( \partial/\partial z_j, \partial/\partial \bar{z}_j \) defined that way cannot be viewed as partial derivatives without some precaution. Strictly speaking, \( \partial/\partial z_j \) can be viewed as a partial derivative — with respect to the holomorphic coordinate system \( z_1, \ldots, z_m \) — only as an operator acting on holomorphic functions, on which the \( \partial/\partial z_j \) acts trivially. On the other hand, the \( \partial/\partial \bar{z}_j \), \( \partial/\partial \bar{z}_j \) can be viewed as partial derivatives, with respect to the a priori fictitious coordinate system \( z_1, \ldots, z_m, \bar{z}_1, \ldots, \bar{z}_m \), if functions on \( U \) are (mentally) assumed to be the restrictions to \( U \) of holomorphic functions defined on some complexification of \( U \), on which \( z_1, \ldots, z_m, \bar{z}_1, \ldots, \bar{z}_m \) becomes a genuine holomorphic coordinate system.

When \( J \) is integrable, the complex vector bundle \( T^{1,0}M \) admits a natural holomorphic structure determined by decreeing that, for any local system of holomorphic coordinates \( z_1, \ldots, z_m \) as above, the corresponding sections \( \partial/\partial z_1, \ldots, \partial/\partial z_m \) of \( T^{1,0}M \) are holomorphic. The corresponding Cauchy-Riemann operator \( \partial - \bar{\partial} \) — cf. (1.6.21) — is defined by \( \partial Z = \sum_{j=1}^m \partial Z_j \otimes \partial/\partial z_j \), if \( Z = \sum_{j=1}^m Z_j \partial/\partial z_j \), and can then be rewritten as

\[
\bar{\partial}Y Z = [Y^{0,1}, Z]^{1,0},
\]
for any section $Z$ of $T^{1,0}M$ and any (real) vector field $Y$.

Via the complex vector bundle isomorphism $X \mapsto Z = X^1 = \frac{1}{2}(X - iJX)$ form $(TM, J)$ to $T^{1,0}M$ and its inverse $Z \mapsto X = Z + \bar{Z}$ from $T^{1,0}M$ to $TM$, this holomorphic structure can be regarded as a holomorphic structure defined on $(TM, J)$ itself, whose Cauchy-Riemann operator then reads

$$\partial_Y X = 2 \mathfrak{Re} (|Y^{0,1}, X^{0,0}|^{1,0}),$$

for any two real vector fields $X, Y$. We then have:

$$\bar{\partial}_Y X = -\frac{1}{2} J(\mathcal{L}_X J)(Y),$$

for all (real) vector fields $X, Y$ (see Section 9.3 for a generalization of this expression in the non-integrable case).

A (real) vector field $X$ is thus holomorphic as a section of $(TM, J)$ equipped with this holomorphic structure — and will then be called a (real) holomorphic vector field, cf. Section 1.23 — if and only if $\mathcal{L}_X J = 0$. From the foregoing, we infer that $X$ is a (real) holomorphic vector field if and only if the $(1,0)$-part $Z = X^{1,0}$ of $X$ is holomorphic in the natural sense, i.e. $Z = \text{loc} \sum_{j=1}^m Z_j \partial/\partial z_j$, where the components $Z_j$ are holomorphic functions of the holomorphic coordinates $z_j$.

For a general almost hermitian structure $(g, J, \omega)$ on $M$, the complex vector bundle $(TM, J)$ comes equipped with a hermitian inner product $h$, defined as follows: the euclidean inner product $(\cdot, \cdot)$ extends to a $\mathbb{C}$-bilinear inner product on $TM \otimes \mathbb{C}$ relatively to which both $T^{1,0}M$ and its conjugate $T^{0,1}M$ are isotropic; together with the natural real structure, this determines a hermitian inner product, $h$, on $TM \otimes \mathbb{C}$ defined by $h(Z_1, Z_2) = (Z_1, \bar{Z}_2)$, then, by restriction, a hermitian inner product on $T^{1,0}M$ and a hermitian inner product on $(TM, J)$ via the above identification $(TM, J) = T^{1,0}M$. The resulting hermitian inner product on $(TM, J)$, still denoted by $h$, is related to $g$ and $\omega$ by $h(X, Y) = \frac{1}{2} (g(X, Y) - i\omega(X, Y))$, for any $X, Y$ in $TM$. Moreover, for any almost-complex structure $J$, $(TM, J)$ admits a canonical Cauchy-Riemann operator, namely the operator determined by (1.8.2), which is (1.8.3) if $J$ is integrable and can be expressed by (9.3.1)-(9.3.2)-(9.3.11) in the general case. The corresponding Chern connection — cf. Proposition 1.6.1 — is called the canonical Chern connection of $(M, g, J, \omega)$ and the corresponding covariant derivative is denoted by $\nabla$. We then have:

**Proposition 1.8.1.** For any almost hermitian manifold $(M, g, J, \omega)$, the canonical Chern connection and the Levi-Civita connection coincide if and only if the structure is Kähler.

**Proof.** If the almost hermitian structure is Kähler, then $J$ is integrable and the Levi-Civita connection $D$ preserves $J$ — it is then a $\mathbb{C}$-linear connection — and preserves $g$, hence the hermitian inner product $h = \frac{1}{2}(g - i\omega)$; moreover, a simple computation of the rhs of (1.8.3) shows that $-\frac{1}{2} J(\mathcal{L}_X J)(Y) = -\frac{1}{2} J[X, JY] - \frac{1}{2} [X, Y] = \frac{1}{2} (D_Y X + JD_JY X)$; in the Kähler case, (1.8.3) then reads:

$$\bar{\partial} = D^{0,1};$$
from the uniqueness of the Chern connection $\nabla$, we infer that $D = \nabla$. Conversely, if $\nabla = D$, then $DJ = \nabla J = 0$, meaning that the structure is Kähler. \hfill $\square$

1.9. The type of exterior forms

The complexified cotangent bundle also splits as a direct sum $T^*M \otimes \mathbb{C} = \Lambda^{1,0}M \oplus \Lambda^{0,1}M$, where $\Lambda^{1,0}M$, resp. $\Lambda^{0,1}M$, denotes the vector bundle of complex covectors $\alpha$ such that $\alpha(JX) = i\alpha(X)$, resp. $\alpha(JX) = -i\alpha(X)$. With respect to the action of $J$ on $T^*M = \Lambda^1M$ defined in Section 1.2 and extended into a $\mathbb{C}$-linear action of $J$ on $\Lambda^1M \otimes \mathbb{C}$, $\Lambda^{1,0}M$ and $\Lambda^{0,1}M$ are then the eigen subbundles relative to the eigenvalues $-i$ and $i$ respectively. We deduce the following isomorphism of complex vector bundles:

\[ \Lambda^r \Lambda^s \mathbb{C} \cong \Lambda^{r+s} \mathbb{C}, \]  

where $\Lambda^{r,0}M = \Lambda_C(\Lambda^{1,0}M)$ (for any complex vector space or complex vector bundle $E$, we agree that $\Lambda_C(E)$ denotes the corresponding full exterior algebra with respect to $\mathbb{C}$); similarly, $\Lambda^{0,s}M = \Lambda_C(\Lambda^{0,1}M)$; we then have $\Lambda^{r,0}M = \oplus_{m=0}^{m=r} \Lambda^{r,0}M$ and $\Lambda^{0,s}M = \oplus_{s=0}^{m=s} \Lambda^{0,s}M$, where $\Lambda^{r,0}M$ is naturally identified with the bundle of complex-valued $r$-forms such that $\phi(JX_1, \ldots, X_r) = i\phi(X_1, \ldots, X_r)$ for any complex vector fields $X_1, \ldots, X_r$; similarly, elements $\psi$ of $\Lambda^{0,s}M$ are complex valued $s$-forms such that $\psi(JX_1, \ldots, X_s) = -i\psi(X_1, \ldots, X_s)$. Elements of $\Lambda^{r,0}M$, resp. of $\Lambda^{0,s}M$, are called (complex) exterior forms of type $(r,0)$, resp. of type $(0,s)$; more generally, elements of $\Lambda^{r,s}M := \Lambda^{r,0}M \otimes \Lambda^{0,s}M$ are called (complex) exterior forms of type $(r,s)$; each (complex) $p$-form $\psi$ can then be written in a unique way as $\psi = \sum_{r+s=p} \psi^{r,s}$, where $\psi^{r,s}$ denotes the component of type $(r,s)$ of $\psi$. To get an easy characterization of elements of $\Lambda^{r,s}M$ in $\Lambda^{r+s}M \otimes \mathbb{C}$, it is convenient to introduce the operator $C$ defined by

\[ (C\psi)(X_1, \ldots, X_p) = -\sum_{j=1}^{p} \psi(X_1, \ldots, JX_j, \ldots, X_p), \]  

for any complex valued $p$-form $\psi$. It is easily checked that $\psi$ is of type $(r,s)$, with $r+s = p$, if and only if $C\psi = (s-r)i\psi$, and we then have: $J\psi = i^{s-r}\psi$.

If $J$ is integrable, and $z_1, \ldots, z_m$ is a local system of holomorphic coordinates defined on some open set $U$, the $dz_j$, resp. $d\bar{z}_j$, are 1-forms of type $(1,0)$ and form a frame of $\Lambda^{1,0}M$ on $U$; similarly, the $d\bar{z}_j$ form a frame of $\Lambda^{0,1}M$: it follows that the $dz_{j_1} \wedge \ldots \wedge dz_{j_r} \wedge d\bar{z}_{k_1} \wedge \ldots \wedge d\bar{z}_{k_s}$, with $j_1 < \ldots < j_r$ and $k_1 < \ldots < k_s$ form a frame of $\Lambda^{r,s}M$ on $U$ for any $0 \leq r, s \leq m$. For each $r = 0, 1, \ldots, m$, $\Lambda^{r,0}M$ has a natural holomorphic structure determined by requesting that the $dz_{j_1} \wedge \ldots \wedge dz_{j_r}$ form a holomorphic frame for any holomorphic coordinate system $z_1, \ldots, z_m$, cf. Section 1.6. In particular, $\Lambda^{1,0}M$ — which, as a complex vector bundle, is isomorphic to $(T^*M, -J)$ — is the (holomorphic) $\mathbb{C}$-dual of the (holomorphic) tangent bundle $(TM, J) = T^{1,0}M$, whereas $\Lambda^{m,0}M$ is the so-called canonical line bundle of $(M, J)$, which will be also denoted by $K_M$.

**Remark 1.9.1.** A real exterior $p$-form $\psi$ cannot be of type $(r,s)$ except if $r = s$ and $\psi$ is then $J$-invariant. In these notes, unless otherwise specified,
Λ^{r,r}M will be implicitly understood as the bundle of real exterior forms of type $(r,r)$.

### 1.10. Riemannian and symplectic Hodge operators

In general, the **riemannian Hodge operator** $\ast$ of an oriented riemannian manifold $(M, g)$ of dimension $n$ is defined by

\begin{equation}
\ast \psi = \psi^g \\
\end{equation}

for each $p$-form $\psi$, where the inner product is understood as follows: whenever $\psi = \alpha_1 \wedge \ldots \wedge \alpha_p$, then $\psi^g = \alpha_1^g \wedge \ldots \wedge \alpha_p^g$ and $(\psi^g)_g(X_{p+1}, \ldots, X_n) = g(\alpha_1^g, \ldots, \alpha_p^g, X_{p+1}, \ldots, X_n)$. Alternatively,

\begin{equation}
\psi_1 \wedge \ast \psi_2 = (\psi_1, \psi_2)_g,
\end{equation}

for all $p$-forms $\psi_1, \psi_2$. The Hodge operator $\ast$ can be defined on multi-vectors as well and is then conveniently determined by

\begin{equation}
\ast (e_1 \wedge \ldots \wedge e_p) = e_{p+1} \wedge \ldots \wedge e_n,
\end{equation}

for any $p$ and any orthonormal, positively oriented basis at any point of $M$. It readily follows that $\ast$ is a (pointwise) isometry and that:

\begin{equation}
\ast^2 \psi = (-1)^p \psi, \quad \text{if } n \text{ is even},
\end{equation}

\begin{equation}
\ast^2 \psi = \psi, \quad \text{if } n \text{ is odd},
\end{equation}

for any $p$-form $\psi$.

In the almost hermitian context, recall — see (1.2.6) — that the action of the almost-complex structure $J$ has been extended to the space of $p$-forms, for any $p$; we then have:

**Lemma 1.10.1.** The rieamnnian Hodge operator $\ast$ and the operator $J$ acting on exterior forms commute.

**Proof.** Let $x$ be any point of $M$ and $\{e_1, \ldots, e_n\}$ be any orthonormal, positively oriented basis of $T_xM$. Let $\psi$ be any $(n-p)$-form on $M$. Then $\ast \psi$ is a $p$-form, whose value at $x$ is entirely defined by its action on any $p$ elements of this basis, which we can chose equal to $e_1, \ldots, e_p$, and we then have $(\ast \psi)(e_1, \ldots, e_p) = \psi(e_{p+1}, \ldots, e_n)$. It follows that $(J \ast \psi)(e_1, \ldots, e_p) = \psi(J^{-1} e_{p+1}, \ldots, J^{-1} e_n)$. Now, $J^{-1} e_1, \ldots, J^{-1} e_n$ is again an orthonormal, positively oriented basis of $T_xM$ (as for any invertible skew symmetric linear operator, the determinant of $J$ is positive, hence equal to $+1$); we then likewise get $(\ast J \psi)(e_1, \ldots, e_p) = \psi(J^{-1} e_{p+1}, \ldots, J^{-1} e_n)$. \hfill $\square$

The **symplectic Hodge operator** is similarly defined by

\begin{equation}
\ast_\omega \psi = \psi^\omega \wedge \omega_g,
\end{equation}

where the operators $\ast_\omega : T^*M \rightarrow TM$ and its inverse $\ast_\omega : TM \rightarrow T^*M$ have been defined in Section 1.2. We thus have:

\begin{equation}
\ast_\omega = J \ast = \ast J,
\end{equation}

so that

\begin{equation}
(\ast_\omega)^2 = 1,
\end{equation}
(by Lemma 1.10.1, $J$ and * commute and, when acting on $p$-forms, are both of square $(-1)^p$, as $n = 2m$ is even).

The following identity is easily checked:

\[ * \frac{\omega^r}{r!} = * \frac{\omega^r}{r!} = \frac{\omega^{m-r}}{(m-r)!}, \]

for each $r = 0, \ldots, m$.

For a general riemannian manifold $(M, g)$, the codifferential $\delta$, acting on exterior forms, is defined as the formal adjoint of $d$, i.e. is determined by

\[
\langle d\phi, \psi \rangle = \langle \phi, \delta \psi \rangle, \quad \text{for any (} p-1 \text{-form } \phi \text{ and any } p\text{-form } \psi, \text{ both compactly supported for} \ (1.10.9) \text{ to make sense. Now, by (1.10.2), we have that} \]

\[
(d\phi, \psi)_v g = d\phi \wedge * \psi = d(\phi \wedge * \psi) + (-1)^p \phi \wedge d(*\psi) = d(\phi \wedge * \psi) + (-1)^p \phi \wedge *(-1) d(*\psi) = d(\phi \wedge * \psi) + (\phi, (-1)^p *(-1) d * \psi)_v g.
\]

Since $d(\phi \wedge * \psi)$ integrates to zero on $M$, we eventually get

\[
\delta \psi = - * d * \psi, \quad \text{if } n \text{ is even, and} \]

\[
\delta \psi = (-1)^p * d * \psi, \quad \text{if } n \text{ is odd, for any } p\text{-form } \psi. \]

From the above computation, we then get the pointwise adjunction formula:

\[
(d\phi, \psi) - (\phi, \delta \psi) = * d(\phi \wedge * \psi) = (-1)^p \delta(\phi \wedge * \psi) = - \delta \alpha,
\]

for any $(p-1)$-form $\phi$ and any $p$-form $\psi$, where $\alpha$ is the 1-form defined by $\alpha(X) = \langle X^\flat \wedge \phi, \psi \rangle = \langle \phi, X \cdot \psi \rangle$.

In terms of the Levi-Civita connection $D$, the codifferential $\delta$ admits the following expression: $\delta \psi = - \sum_{j=1}^n D_{e_j} \psi(e_j, \cdot, \cdot, \cdot)$, for any auxiliary orthonormal frame $\{e_1, \ldots, e_n\}$, i.e.

\[
\delta \psi = - \sum_{i=1}^n e_{i \cdot} D_{e_i} \psi.
\]

For any vector bundle $E$ over $M$, equipped with a fiberwise inner product — euclidean or hermitian according as $E$ is real or complex — and with a linear connection $\nabla$ which preserves the inner product, we can similarly define a codifferential $\delta^\nabla$ relative to $\nabla$ as the formal adjoint of $d^\nabla$. If $D^\nabla$ denotes the connection induced by $\nabla$ and the Levi-Civita connection $D$ acting on the space of $E$-valued exterior forms, cf. Section 1.6, we then have

\[
\delta^\nabla \Phi = - \sum_{i=1}^n e_{i \cdot} D^\nabla_{e_i} \Phi.
\]
for any \( E \)-valued exterior form \( \Phi \) and any auxiliary orthonormal frame \( \{ e_1, \ldots, e_n \} \). For decomposed \( E \)-valued exterior forms \( \Phi = s \otimes \psi \), we then have

\[
\delta \nabla (s \otimes \psi) = - \sum_{i=1}^{n} e_i \mathcal{J} e_i s \otimes \psi + s \otimes \delta \psi.
\]

**Remark 1.10.1.** The rhs of (1.10.13) makes sense when we substitute any \( p \)-multilinear form, i.e., any section \( \psi \) of \( \otimes^p T^* M \), for any \( p \); then (1.10.13) can be viewed as a definition of \( \delta \psi \). For any \((p-1)\)-multilinear form \( \phi \) and any \( p \)-multilinear form \( \psi \), both compactly supported, we then have

\[
\langle D\phi, \psi \rangle = \langle \phi, \delta \psi \rangle,
\]

where the global inner product is relative to the tensorial inner product in \( \otimes^p T^* M \), cf. Section 1.3. This means that \( \delta \) then coincides with the adjoint, \( D^* \), of the Levi-Civita covariant derivative \( D \).

When applied to an exterior form, \( \delta \) is the adjoint of the exterior derivative \( d \) as a section of the bundle \( \Lambda^*_M \) of exterior forms, equipped with the specific inner product of exterior forms — cf. Section 1.3 — and the adjoint of the Levi-Civita covariant derivative \( D \) as a section of \( \otimes^p T^* M \), equipped with the tensorial inner product. The coherence of the two interpretation is then guaranteed by (1.3.5).

**Remark 1.10.2.** The **divergence**, \( \operatorname{div} X \), of a vector field is defined by

\[
\mathcal{L}_X v = \langle \phi, \delta \psi \rangle,
\]

we thus have \( \operatorname{div} X = - \delta X^\flat \).

### 1.11. Twisted exterior differential and twisted codifferential

The **twisted exterior differential** \( d^c \) is defined by

\[
d^c \psi = J d J^{-1} \psi,
\]

for any \( p \)-form \( \psi \), where \( J^{-1} = (-1)^p J \) is the inverse of \( J \) acting on \( p \)-forms. In particular, we have: \( d^c f = J df \), i.e., \( (d^c f)(X) = -df(JX) \), for each function \( f \).

The adjoint of \( d^c \), denoted by \( \delta^c \), is the **twisted codifferential**, defined by:

\[
\delta^c = J \delta J^{-1}
\]

We then have:

**Proposition 1.11.1.** The twisted codifferential \( \delta^c \) is entirely determined by the symplectic form \( \omega \) only. More precisely, for any \( p \)-form \( \psi \), we have:

\[
\delta^c \psi = (-1)^{p+1} \ast \omega \ast d \ast \omega \psi.
\]

Alternatively, for any linear connection \( \tilde{D} \) on \( TM \) which is torsion-free and preserves \( \omega \), we have:

\[
\delta^c \psi = - \sum_{i=1}^{n} \tilde{e}_i \mathcal{J} \tilde{e}_i \psi,
\]

for any local frame \( \{ e_1, \ldots, e_n \} \) of \( TM \), where the symplectic dual frame \( \{ \tilde{e}_1, \ldots, \tilde{e}_n \} \) is determined by \( \omega(e_i, \tilde{e}_j) = \delta_{ij} \).
Proof. The first assertion follows from any one of the explicit two expressions (1.11.3) or (1.11.4) of $\delta^c$, whereas (1.11.3) is a direct consequence of the definition (1.11.2) of $\delta^c$, by using (1.10.10) and (1.10.6). From (1.3.5)—which, as already observed, holds when, in the rhs, the Levi-Civita connection $D^g$ is replaced by any torsion-free linear connection, in particular by $\bar{D}$—we infer that:

$$
\delta^c \psi = -(-1)^p *_\omega d *_\omega \psi
$$

$$
= -(-1)^p *_\omega \sum_{i=1}^n e_i^* \wedge \bar{D}_{e_i}(*_\omega \psi)
$$

$$
= -(-1)^p \sum_{i=1}^n *_\omega(e_i^* \wedge *_\omega \bar{D}_{e_i} \psi)
$$

$$
= -\sum_{i=1}^n \tilde{e}_i \bar{D}_{e_i} \psi.
$$

□

Remark 1.11.1. By defining $\langle \psi_1, \psi_2 \rangle_\omega := \int_M \psi_1 \wedge *_\omega \psi_2,$ for any two $p$-forms $\psi_1, \psi_2$, we get a non-degenerate pairing on the space of (compactly supported) real $p$-forms, determined by the symplectic form $\omega$; when $\omega = g(J\cdot, \cdot)$, this symplectic pairing can be alternatively defined by $\langle \psi_1, \psi_2 \rangle_\omega = \langle \psi_1, J\psi_2 \rangle$: it is anti-symmetric if $p$ is odd, symmetric if $p$ is even and, in the latter case, is never definite. We then have

$$
d^* \omega = -\delta^c,
$$

where $d^* \omega$ denotes the symplectic adjoint of $d$, defined by the symplectic adjunction relation $\langle d\phi, \psi \rangle_\omega = \langle \psi, d^* \omega \psi \rangle_\omega$. We similarly get:

$$
(\delta^c)^* \omega = d.
$$

Notice that the operation of symplectic adjunction is anti-involutive in this case.

Remark 1.11.2. In the Kähler setting, we may choose the Levi-Civita connection $D^g$ in place of $\bar{D}$ in the rhs of (1.11.4); moreover, if $\{e_1, \ldots, e_n\}$ is orthonormal, the symplectic dual frame is clearly the frame $\{J e_1, \ldots, J e_n\}$; (1.11.4) then reads

$$
\delta^c \psi = -\sum_{i=1}^n J e_i \wedge D^g_{e_i} \psi.
$$

We easily deduce from (1.3.5) the following expression of $d^c$:

$$
d^c \phi = \sum_{i=1}^n J e_i^* \wedge D^g_{e_i} \phi,
$$

for any frame $\{e_1, \ldots, e_n\}$ of $TM$; this expression is in fact valid on any complex manifold $(M, J)$ when $J$ is integrable and $D^g$ is replaced by any torsion-free linear connection which preserves $J$ (easy verification).
The twisted exterior differential \( d^c \) is entirely determined by \( J \) and can be defined for any almost-complex structure \( J \). We thus obtain the following alternative integrability criterion for \( J \):

**Proposition 1.11.2.** The almost-complex \( J \) is integrable if and only if \( d^c \) and \( \delta^c \) anticommute, i.e. if and only if

\[
\delta^c + \delta^c d = 0.
\]

(1.11.9)

This happens if and only if \( \delta^c f + \delta^c df = 0 \) or, equivalently, if and only if \( \delta^c f \) is \( J \)-invariant, for any function \( f \).

**Proof.** The operator \( \delta^c + \delta^c d \), acting on exterior forms, is a derivation for the exterior product, i.e. satisfies \( (\delta^c + \delta^c d)(\phi \wedge \psi) = (\delta^c + \delta^c d)\phi \wedge \psi + \phi \wedge (\delta^c + \delta^c d)\psi \) for any two exterior forms \( \phi, \psi \) — easy verification — and obviously commutes with \( d \); it is then entirely determined by its action on functions. By using (1.3.3), for any (real) function \( f \) and any two vector fields \( X, Y \), we get

\[
(\delta^c f)_{X,Y} = X \cdot \delta^c f(Y) - Y \cdot \delta^c f(X) - d^c f([X,Y]) = -X \cdot (JY \cdot f) + Y \cdot (JX \cdot f) + J[X,Y] \cdot f,
\]

whereas

\[
(\delta^c df)_{X,Y} = -(\delta^c f)_{JX,JY} = -JX \cdot (Y \cdot f) + JY \cdot (X \cdot f) - J[JX,JY] \cdot f;
\]

by comparing with (1.1.2), we infer

\[
(\delta^c f + \delta^c df)_{X,Y} = (\delta^c f - J(\delta^c f))_{X,Y} = 4\delta^c f(N(X,Y))
\]

where, we recall, \( N \) denotes the Nijenhuis tensor of \( J \) defined by (1.1.2). It readily follows that \( \delta^c + \delta^c d \) acts trivially on functions, hence on the whole space of exterior forms, if and only if \( N \) is identically zero. The last assertion again readily follows from (1.11.10).

**Remark 1.11.3.** When \( J \) is integrable, we have that

\[
d = \partial + \bar{\partial},
\]

(1.11.11)

where \( \partial, \bar{\partial} \) are determined as follows: for any form \( \psi \) of type \((r,s) \), \( \partial \psi \) and \( \bar{\partial} \psi \) are defined by:

\[
\partial \psi = (d\psi)^{r+1,s}, \quad \bar{\partial} \psi = (d\psi)^{r,s+1}
\]

(1.11.12)

(it easily follows from (1.3.3) that, when \( J \) is integrable, i.e. when \( T^{1,0}M \) and \( T^{0,1}M \) are closed for the bracket, \( d\psi \) has no other components than the above two). We then have

\[
d^c = i(\bar{\partial} - \partial),
\]

(1.11.13)

hence

\[
\partial = \frac{1}{2}(d + id^c), \quad \bar{\partial} = \frac{1}{2}(d - id^c),
\]

(1.11.14)

\[
\partial^2 = \bar{\partial}^2 = \partial \bar{\partial} + \bar{\partial} \partial = 0,
\]

(1.11.15)

and

\[
\delta^c d^c = 2i\partial \bar{\partial}.
\]

The operator \( \partial^* \), \( \bar{\partial}^* \) are defined as the formal adjoints of \( \partial \), \( \bar{\partial} \) with respect to the global hermitian inner product defined on the space of complex valued exterior forms, cf. Section 1.8; we then have

\[
\partial^* = \frac{1}{2}(\delta - i\delta^c), \quad \bar{\partial}^* = \frac{1}{2}(\delta + i\delta^c),
\]

(1.11.17)
The definition of the twisted differential $d^c$ readily extends to $E$-valued exterior forms over an almost-complex manifold $(M,J)$ for any (real) vector bundle $E$ over $M$ equipped with a linear connection $\nabla$. We then define $d^c$, $\nabla$ by:

$$d^c, \nabla \Phi = J d \nabla J - 1 \Phi, \quad \text{for any } E\text{-valued } p\text{-form } \Phi; \quad \text{here } \Phi \text{ is viewed as an element of } E \otimes \Lambda^p M \text{ and the action of } J \text{ is simply induced by the action of } J \text{ on } \Lambda^p M \text{ defined by } (1.2.6).$$

For any decomposed element $s \otimes \phi$, where $s$ is a section of $E$ and $\phi$ a scalar $p$-form, we then have

$$(1.11.24) \quad (d^c \nabla d^c + d^c d^c) \Phi = -C(R^\nabla) \wedge \Phi,$$
for any $E$-valued exterior form $\Phi$, where $C$ denotes the operator defined by (1.9.2), i.e.

\[(1.11.25)\]
\[C(R^\nabla)_{X,Y} = -R^\nabla_{JX,Y} - R^\nabla_{X,JY}.\]

In particular, $d^\nabla d^c^\nabla + d^c^\nabla d^\nabla = 0$ if and only if $R^\nabla$ is of type $(1,1)$.

**Proof.** It is sufficient to check (1.11.24) when $\Phi = s \otimes \phi$ is a decomposed $E$-valued $p$-form. From (1.11.19), we easily infer

\[
(d^\nabla d^c^\nabla + d^c^\nabla d^\nabla)(s \otimes \phi) = \sum_{i,j=1}^n R^\nabla_{e_i,e_j} s \otimes e^*_j \wedge J e^*_i \wedge \phi
\]

\[
+ s \otimes (d^\nabla + d^c^\nabla)\phi,
\]

where $(d^c^\nabla + d^\nabla)\phi = 0$, as $J$ is integrable, whereas the curvature term in the rhs is clearly equal to $-C(R^\nabla)$. For 2-forms, the kernel of $C$ is the space of forms of type $(1,1)$; the last assertion then follows from (1.11.24).

**Remark 1.11.4.** When $E = (E,h)$ is a hermitian holomorphic vector bundle over a complex manifold $(M,J)$ and $\nabla$ is the corresponding Chern connection — cf. Section 1.6 — then the extension of the Cauchy-Riemann operator $\partial^E$ to the $E$-valued exterior forms of type $(0,q)$, as defined by (1.6.19), coincides with the action of the operator $\frac{i}{2}(d^\nabla - id^c^\nabla)$. It can then be extended to the whole space of $E$-valued exterior forms, of any type, by simply setting:

\[
(1.11.26)\]
\[\partial^E = \frac{1}{2}(d^\nabla - id^c^\nabla).
\]

We similarly extend the action of $\partial^\nabla = \nabla^{1,0}$ to the space of $E$-valued exterior forms by setting:

\[
(1.11.27)\]
\[\partial^\nabla = \frac{1}{2}(d^\nabla + id^c^\nabla).
\]

Since the Chern curvature $R^\nabla$ is $J$-invariant, $d^\nabla d^c^\nabla + d^c^\nabla d^\nabla = 0$ — cf. Proposition 1.11.3 — whereas $(d^c^\nabla)^2 \Phi = -J(R) \wedge \Phi = -R \wedge \Phi = (d^\nabla)^2 \Phi$. This implies $(\partial^E)^2 = \frac{1}{4}((d^\nabla)^2 - (d^c^\nabla)^2 - i(d^\nabla d^c^\nabla + d^c^\nabla d^\nabla)) = 0$ and, similarly, $(\partial^\nabla)^2 = 0$.

**Remark 1.11.5.** For any non-negative integers $p,q$, the space $\Gamma(E \otimes \Lambda^{p,q}M)$ of $E$-valued exterior forms of type $(p,q)$ is naturally identified with the space $\Gamma((E \otimes \Lambda^{p,0}M) \otimes \Lambda^{0,q}M)$ of $E \otimes \Lambda^{p,0}M$-valued exterior forms of type $(0,q)$, where $\Lambda^{p,0}M$ is viewed as a holomorphic vector bundle, via the natural holomorphic structure of the tangent space $TM \cong T^{1,0}M$, cf. Section 1.6. Also recall that when $\Lambda^{p,0}M$ is equipped with the hermitian inner product induced by any Kähler metric on $M$, the corresponding Chern connection is the (induced) Levi-Civita connection. It is then easily checked that the extension of the operator $\partial^E$ defined by (1.11.26) on $\Gamma(E \otimes \Lambda^{p,q}M)$ coincides with the natural extension of the holomorphic structure $\partial^E \otimes \Lambda^{p,0}M$ of the complex vector bundle $E \otimes \Lambda^{p,0}$ to $\Gamma((E \otimes \Lambda^{p,0}M) \otimes \Lambda^{0,q}M)$, as defined by (1.6.19).
1.12. J-invariant 2-forms, hermitian operators and Chern forms

Any (real) 2-form \( \psi \) can be regarded as a skew-symmetric endomorphism of \( TM \), still denoted by \( \psi \), via the metric \( g \): \( \psi(X, Y) = g(\psi(X), Y) \). The action of \( J \) defined in Section 1.2 has then the following expression:

\[ \psi \rightarrow J \circ \psi \circ J^{-1}, \]

so that \( \psi \) is \( J \)-invariant if and only if the corresponding operator commutes with \( J \). The vector bundle of \( J \)-invariant elements of \( \Lambda^2 M \) coincides with the bundle of (real) 2-forms of type \((1,1)\) and is therefore denoted by \( \Lambda^{1,1} M \). Any element \( \psi \) of \( \Lambda^{1,1} M \) can be viewed as a skew-hermitian endomorphism of \( (TM, J, h) \) and thus be written as

\[ \psi = J \Psi, \]

where \( \Psi \) is a hermitian endomorphism of \( (TM, J, h) \).

Let \( P(t) = P^\Psi(t) = \sum_{r=0}^{m} (-1)^r \sigma_r(\phi) t^{m-r} \) be the characteristic polynomial of \( \Psi \), where, for \( r = 1, \ldots, m \), \( \sigma_r(\phi) \) denotes the \( r \)-th elementary symmetric function of the \( m \) (real) eigenvalues of \( \Psi \) at each point, and \( \sigma_0(\phi) \equiv 1 \).

Each \( \sigma_r \) is a homogeneous real polynomial function of degree \( r \) defined on \( \Lambda^{1,1} M \). The corresponding polarized form, still denoted by \( \sigma_r \), is then a \( r \)-multilinear form defined on \( \Lambda^{1,1} M \).

The linear form \( \sigma_1 \) is also called the trace of \( \psi \) with respect to \( \omega \), denoted \( \text{tr}_\omega(\psi) \) or, simply, \( \text{tr}(\psi) \), whereas \( \sigma_m \) coincides with the pfaffian, defined by

\[ \sigma_m = \text{pf}(\psi) \omega^m. \]

It follows that \( P(t) = \text{pf}(t\omega - \psi) \).

By developing this expression in powers of \( t \), we thus get the following expression for the polarized \( \sigma_r \):

\[ \sigma_r(\psi_1, \ldots, \psi_r) \omega^m = \frac{m!}{(m-r)! r!} \psi_1 \wedge \ldots \wedge \psi_r \wedge \omega^{m-r}, \]

for any sections \( \psi_1, \ldots, \psi_r \) of \( \Lambda^{1,1} M \).

Denote by \( L_\omega \) and \( \Lambda_\omega \), or simply \( L \) and \( \Lambda \) if \( \omega \) is understood, the adjoint pair of operators acting on exterior forms, defined by

\[ L(\psi) = \omega \wedge \psi, \quad \Lambda(\psi) = \omega^\# \psi = \omega^{\sharp} \omega. \]

Then, by using (1.10.8) we easily check that (1.12.2) can also be written as

\[ \sigma_r(\psi_1, \ldots, \psi_r) = \frac{1}{(r!)^2} \Lambda^r(\psi_1 \wedge \ldots \wedge \psi_r). \]

In particular, the quadratic form \( \sigma_2 \) can be written as

\[ \sigma_2(\psi_1, \psi_2) = \frac{1}{2}(\text{tr} \psi_1 \text{tr} \psi_2 - (\psi_1, \psi_2)). \]

Notice that \( \Lambda^{1,1} M \) can be regarded as the adjoint bundle of the Kähler structure, i.e. the vector bundle associated to the adjoint action of the unitary group \( U(m) \) on its Lie algebra \( u(m) \), and that the \( \sigma_r \) can be viewed as \( U(m) \)-invariant polynomial functions defined on \( u(m) \). The corresponding characteristic class is then the \( r \)-th Chern class of \( (TM, J) \) (more details in Appendix A).
**1.13. The Hodge-Lepage decomposition**

The operators $L, \Lambda$ defined in Section 1.12 are related to each other by

$$\Lambda = * L *^{-1} = *_{\omega} L *_{\omega}. \tag{1.13.1}$$

This can be easily checked by using the following explicit expressions of $L, \Lambda$:

$$L \psi = \frac{1}{2} \sum_{i=1}^{n} e_i \wedge J e_i \wedge \psi, \quad \Lambda \psi = \frac{1}{2} \sum_{i=1}^{n} J e_{i,j} J e_{i,j} \psi, \tag{1.13.2}$$

for any auxiliary (local) orthonormal frame $\{e_1, \ldots, e_{2m}\}$, and the general riemannian identity:

$$X \wedge \psi = * (X^p \wedge * \psi), \tag{1.13.3}$$

for any exterior form $\psi$ and any vector field $X$, on any even-dimensional riemannian manifold (on a manifold of dimension $n$, the right hand side must be multiplied by $(-1)^{n(p-1)}$ whenever $\psi$ is of degree $p$), whose easy verification is left to the reader. Recall that in (1.13.1), $*^{-1} = (-1)^p$, whereas $*_{\omega} = *_{\omega}^{-1}$, cf. (1.10.4)–(1.10.7).

It easily follows from (1.12.3) that

$$[\Lambda, L] = H, \tag{1.13.4}$$

where $H$ is defined by

$$H(\psi) = (m-p) \psi, \tag{1.13.5}$$

for any $p$-form $\psi$. We also easily check that

$$[H, \Lambda] = 2 \Lambda, \quad [H, L] = -2L. \tag{1.13.6}$$

These are the commutation rules of the standard generators of the (complex) Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, via the identifications:

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.13.7}$$

Now, $L, \Lambda, H$ are zero order operators and, upon extending their action to complex exterior forms by $\mathbb{C}$-linearity, can be viewed as algebraic operators acting on the complex vector space $V := \sum_{p=0}^{2m} V^p$, with $V^p = \Lambda^p \mathbb{C}^{2m} \otimes \mathbb{C}$. The kernel of $\Lambda$ acting on $V$ is denoted by $V_0$. The elements of $V_0$ are called primitive. Notice that the degree of a non-zero primitive form cannot be greater than the complex dimension $m$ of $M$. Indeed, if $\psi$ is any primitive form of degree $p$, with $p > m$, then, by (1.13.4), $\Lambda L \psi = (m-p) \psi$, hence $|L \psi|^2 = (m-p) |\psi|^2$, with $(m-p) < 0$, so that $\psi = 0$. By a similar argument, we easily check that the map $L : V^p \to V^{p+2}$ is injective for any $p < m$; indeed, if $\psi$ is any $p$-form, with $p < m$ and $L \psi = 0$, then, $[\Lambda, L] \psi = -L \Lambda \psi = (m-p) \psi$, hence $|\Lambda \psi|^2 = (p-m) |\psi|^2$, with $(p-m) < 0$, so that $\psi = 0$. Equivalently, the map $\Lambda : V^{p+2} \to V^p$ is surjective whenever $p < m$. We thus have

$$V_0 = \bigoplus_{p=0}^{m} V_0^p, \tag{1.13.8}$$

where $V_0^p$ denotes the space of primitive $p$-forms of degree $p$. Moreover, $V_0^0 = V^0$, $V_0^1 = V^1$, whereas, for $p = 2, \ldots, m$, $\dim \mathbb{C} V_0^p = \binom{2m}{p} - \binom{2m}{p-2}$. 

Irreducible (complex) linear representations of \( \mathfrak{sl}(2, \mathbb{C}) \) are indexed by non-negative integers. For each integer \( r \geq 0 \), the corresponding representation space \( S^{(r)} \simeq \mathbb{C}^{r+1} \) is the \( r \)-th (complex) symmetric tensor power of the natural representation space \( S^{(1)} = \mathbb{C}^2 \) and can be described as follows: in \( S^{(r)} \) there is, up to a multiplicative factor, a unique element, \( v_0 \), such that \( \Lambda v_0 = 0 \) and \( H v_0 = r v_r \); then, \( v_0, v_1 = L v_0, v_2 = L^2 v_0, \ldots, v_r = L^r v_0 \) form a basis of \( S^{(r)} \), such that \( H(v_j) = (r - 2j) v_j \), for each \( 0 \leq j \leq r \) (\( v_r \) is a dominant vector, corresponding to the dominant weight \( r \), and \( r, r - 2, \ldots, -r + 2, -r \) are the weights of the representation).

When \( V \) is viewed as a \( \mathfrak{sl}(2, \mathbb{C}) \)-representation space, the isotypic component relative to \( S^{(r)} \), say \( V^{(r)} \), is thus isomorphic to \( S^{(r)} \otimes V_0^{m-r} \) — each element \( \phi \) of \( V^{(r)} \) is then written in a unique way as \( \phi = \sum_{j=0}^r L_j \phi_j \), where \( \phi_j \) is a primitive form of degree \( m - r \) — and we eventually get

\[
V = \sum_{r=0}^m V^{(r)} \cong \sum_{r=0}^m S^{(r)} \otimes V_0^{m-r},
\]

where each \( V_0^{m-r} \) appears as a parameter space, on which the action of \( \mathfrak{sl}(2, \mathbb{C}) \) is trivial, and whose dimension is the multiplicity of \( S^{(r)} \) in \( V \).

Each element of \( V \), in particular each complex-valued \( p \)-form \( \psi \), is expressed in a unique manner as the sum of elements in \( V^{(r)} \). We thus get the Hodge-Lepage decomposition\(^1\) of any (complex) \( p \)-form \( \psi \):

\[
\psi = \sum_{j=\lfloor \frac{p}{2} \rfloor}^{\lfloor \max(0, \frac{m}{p}) \rfloor} L_j \psi_j,
\]

where \( \psi_j \) is a (uniquely defined) primitive form of degree \( p - 2j \) (more details can be found, e.g., in Chapter V, Section 3 of [193]).

**Remark 1.13.1.** It follows from the foregoing that at each point \( x \) of \( M \), any non-zero primitive form \( \phi_0 \) of degree \( p \), together with the \( (m - p) \) exterior forms \( \phi_1 = L \phi_0, \ldots, \phi_{m-p} = L^{m-p} \phi_0 \), form a base of a \( (m - p + 1) \)-dimensional subspace of \( \Lambda^p M \), isomorphic to \( S^{(m-p)} \), so that

\[
L^{m-p+1} \phi_0 = 0,
\]

whereas, by (1.13.4),

\[
\Lambda^r \phi_r = r(m - p - r + 1) \phi_{r-1},
\]

\( r = 1, \ldots, m - p \).

The following identity appears in A. Weil’s book [192] as Théorème 2 of the Chapitre I, and, with a different argument due to Henryk Hecht, as Theorem 3.16 in Chapter V, Section 3 of R. O. Wells’ book [193]. The elementary argument given here is due to A. Moroianu.

**Proposition 1.13.1.** For any primitive \( p \)-form \( \psi \), \( 0 \leq p \leq m \), and for any integer \( r \), \( 0 \leq r \leq m - p \), we have

\[
* L^r \psi = (-1)^{\frac{p(p-1)}{2}} \frac{r!}{(m-p-r)!} L^{m-p-r} J \psi
\]

\(^{1}\) also called Lefschetz decomposition in the literature. This decomposition appears in [192, Chapitre I] as Théorème 3.
or, equivalently, the purely symplectic identity:

\[(1.13.14) \quad \ast \omega L^r \psi = (-1)^{\frac{p(p-1)}{2}} \frac{r!}{(m-p-r)!} L^{m-p-r} \psi.\]

**Proof.** We first notice that (1.13.14) readily follows from (1.13.13) and (1.10.6) — and vice versa — as $J$ obviously commutes with $L$. We then only need to check (1.13.13). By a first easy induction on $r$, it is furthermore sufficient to check (1.13.13) for $r = 0$, i.e. to check

\[(1.13.15) \quad \ast \psi = \frac{(-1)^{p(p-1)}}{(m-p)!} L^{m-p} J \psi,\]

which is the same as

\[(1.13.16) \quad \ast \omega = \frac{(-1)^{p(p-1)}}{(m-p)!} L^{m-p} \psi.\]

Indeed, assuming that $r \geq 1$ and that (1.13.13) holds when $r$ is replaced by $r - 1$, by using (1.13.1) and (1.13.12), we get:

\[
\begin{align*}
&\ast L^r \psi = \Lambda \ast L^{r-1} \psi \\
&= (-1)^{\frac{p(p-1)}{2}} \frac{(r-1)!}{(m-p-r+1)!} \Lambda L^{m-p-r+1} J \psi \\
&\quad + (-1)^{\frac{p(p-1)}{2}} \frac{(r-1)!}{(m-p-r+1)!} r(m-p-r+1) L^{m-p-r} J \psi \\
&\quad + (-1)^{\frac{p(p-1)}{2}} \frac{r!}{(m-p-r)!} L^{m-p-r} J \psi.
\end{align*}
\]

Finally, we check (1.13.15) by induction on the degree of the primitive $p$-form $\psi$, by observing that if, $p \geq 1$, $X \cdot \psi$ is then a primitive $(p - 1)$-form for any vector field $X$ (easy consequence of (1.13.2)). Notice that, for $p = 0$, (1.13.15) reduces to the well-known identity $\ast 1 = \frac{\omega^m}{m!}$. The induction then goes as follows. If $p \geq 1$, for any vector field $X$ we infer from (1.13.1) and (1.13.12) that

\[(1.13.17) \quad \ast (X \cdot \psi) = (-1)^{p-1} X^\flat \wedge \ast \psi,
\]

whereas, by the induction hypothesis applied to the $(p - 1)$-primitive form $X \cdot \psi$, we get

\[(1.13.18) \quad \ast (X \cdot \psi) = (-1)^{\frac{(p-1)(p-2)}{2}} \frac{1}{(m-p+1)!} \omega^{m-p+1} \wedge (JX \cdot J \psi)
\]

\[
\begin{align*}
&\quad + (-1)^{\frac{(p-1)(p-2)}{2}} \frac{1}{(m-p+1)!} [JX \cdot (\omega^{m-p+1} \wedge J \psi) + (m-p+1) X^\flat \wedge \omega^{m-p} \wedge J \psi] \\
&= (-1)^{\frac{(p-1)(p-2)}{2}} \frac{1}{(m-p)!} X \wedge \omega^{m-p} \wedge J \psi
\end{align*}
\]

because of (1.13.11). By comparing the above two expressions of $\ast (X \cdot \psi)$ we obtain that $X \wedge \ast \psi = X \wedge [(-1)^{\frac{p(p-1)}{2}} \frac{1}{(m-p)!} L^{m-p} J \psi]$ for any vector field $X$; this clearly implies (1.13.15). □
Remark 1.13.2. H. Hecht's argument reproduced in Wells' book \cite{193} uses the fact that the above \(\mathfrak{s}(2, \mathbb{C})\)-action on the space of exterior forms can be integrated into a \(\mathbb{C}\)-linear action of the group \(SL(2, \mathbb{C})\) and that the action of the so-called Weyl operator \(w = \exp(\frac{i\pi}{2}(\Lambda + L))\), associated to the element 
\[
\exp\left(\frac{i\pi}{2}(X + Y)\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
of \(SL(2, \mathbb{C})\), is then the same as the action of the symplectic Hodge operator \(*_\omega\) up to some factor; more precisely, it can be checked that
\[
* \omega \psi = i^{p^2 - m} w(\psi),
\]
for any \(p\)-form \(\psi\), cf. Lemma 3.15 and the proof of Theorem 3.16 in \cite{193, Chapter V}.

Remark 1.13.3. The identities in Sections 1.12 and 1.13 are purely algebraic, i.e. involve no derivatives, hence are valid on any almost hermitian manifold.

1.14. Kähler identities

In spite of the name Kähler identities commonly given to identities \((1.14.1)\), these actually hold in the symplectic framework, i.e. for any almost hermitian structure \((g, J, \omega)\) such that \(\omega\) is a symplectic 2-form but \(J\) is not assumed to be integrable, cf. Remark 1.1.2\(^2\).

Proposition 1.14.1. On any symplectic almost hermitian manifold \((M, g, J, \omega)\), the following identities hold:

\[
(1.14.1) \quad [\Lambda, d^c] = \delta, \quad [\Lambda, d] = -\delta^c,
\]
and

\[
(1.14.2) \quad [\delta^c, L] = d, \quad [\delta, L] = -d^c,
\]

where both sides are operators acting on exterior forms and the brackets denote commutators.

Proof. Since \(L\) and \(\Lambda\) (obviously) commute with \(J\), any identity in \((1.14.1)\) implies the other one; similarly, any identity in \((1.14.2)\) implies the other one. Moreover, two of these four identities, namely the second one in \((1.14.1)\) and the first one in \((1.14.2)\), are purely symplectic, i.e. involve operators determined by \(\omega\) only — cf. proposition 1.11.1 — and, by \((1.11.5)-(1.11.6)\), are symplectically adjoint to each other (whereas each identity in \((1.14.1)\) is metrically adjoint to the corresponding one in \((1.14.2)\)). It is then sufficient to check any one of them, say the first identity of \((1.14.2)\), which is an easy consequence of \((1.11.4)\), via the following simple computation (where

\(^{2}\)I personally learned of this fact from R. Bryant many years ago.
\( \tilde{D} \) denotes any auxiliary torsion-free, \( \omega \)-preserving linear connection):

\[
[c^\mathcal{C}, L] \psi = \delta^c(\omega \wedge \psi) - \omega \wedge \delta^c \psi
\]

\[
= \sum_{i=1}^{n} (-\tilde{e}_i \lrcorner \tilde{D}_{e_i} (\omega \wedge \psi) + \omega \wedge \tilde{e}_i \lrcorner \tilde{D}_{e_i} \psi)
\]

\[
= \sum_{i=1}^{n} (-\tilde{e}_i \lrcorner (\omega \wedge \tilde{D}_{e_i} \psi) + \omega \wedge \tilde{e}_i \lrcorner \tilde{D}_{e_i} \psi)
\]

\[
= \sum_{i=1}^{n} e_i^* \wedge \tilde{D}_{e_i} \psi = d\psi,
\]

by using \( \tilde{e}_i \lrcorner \omega = -e_i^* \), which readily follows from the definition of the symplectic dual frame \( \{ \tilde{e}_1, \ldots, \tilde{e}_n \} \), as in Proposition 1.11.1 (here, as usual, \( \{ e_1^*, \ldots, e_n^* \} \) stands for the algebraic dual coframe of the chosen auxiliary frame \( \{ e_1, \ldots, e_n \} \)). \( \square \)

**Proposition 1.14.2.** On any almost-Kähler manifold, the following identities hold:

\[
\delta d^c + d^c \delta = 0, \quad \delta^c d + d\delta^c = 0.
\]

**Proof.** Direct consequence of Proposition 1.14.1: by (1.14.1), \( \delta d^c = [\Lambda, d^c]d^c = -d^c \Lambda d^c \), whereas \( \delta^c d = \delta^c[\Lambda, d^c] = \delta^c \Lambda d^c \); similarly, \( \delta^c d = -[\Lambda, d]d = d\Lambda d \), whereas \( d\delta^c = -d[\Lambda, d] = -d\Lambda d \). Notice that the second identity in (1.14.3) is purely symplectic. \( \square \)

In the context of \( E \)-valued exterior forms, Proposition 1.14.1 remains formally the same:

**Proposition 1.14.3.** For any symplectic almost hermitian manifold \( (M, g, J, \omega) \) and for any vector bundle \( E \), equipped with a fiberwise inner product \( h \) and an \( h \)-compatible linear connection \( \nabla \), the following identities hold:

\[
[\Lambda, d^c \nabla] = \delta^c \nabla, \quad [\Lambda, d \nabla^c] = -\delta^c \nabla, \quad \delta^c \nabla, \quad \delta \nabla^c,
\]

and

\[
[\delta^c \nabla, L] = d^c \nabla, \quad [\delta \nabla, L] = -d^c \nabla,
\]

where both sides are operators acting on \( E \)-valued exterior forms.

**Proof.** The four differential operators \( d^\nabla, d^c \nabla, \delta^c \nabla, \delta \nabla^c \) have been defined in Sections 1.6 and 1.11, whereas operators \( L \) and \( \Lambda \) readily extend to \( E \)-valued forms by: \( L(s \otimes \psi) = s \otimes L\psi \) and \( \Lambda(s \otimes \psi) = s \otimes \Lambda\psi \). In order to check that these identities remain valid at some point \( x \) of \( M \), it is sufficient to compare the actions of these operators on \( \Psi = \sum_{k=1}^{r} s_k \otimes \psi_k \), where \( \{ s_1, \ldots, s_r \} \) is a local frame of \( E \) and the \( s_k \)'s have been chosen in such a way that \( \nabla s_k \) vanishes at \( x \); then, (1.14.4)-(1.14.5) readily follow from (1.14.1)-(1.14.2). \( \square \)
Remark 1.14.1. Proposition 1.14.2 does not extend to $E$-valued exterior forms in such a simple way, as $(d^E)^2$ and $(d^cE)^2$ are not zero in general. We have instead:

\[ (\delta^E d^cE + d^cE \delta^E) \Phi = -\Lambda (JR^\nabla \wedge \Phi) + JR^\nabla \wedge \Lambda \Phi, \]

(1.14.6)

where $J(R^\nabla)$ is defined by $J(R^\nabla)_{X,Y} = R^\nabla_{JX,JY}$. Indeed, from (1.14.5) we infer $\delta^\nabla d^cE + d^cE \delta^\nabla = [\Lambda, d^cE]d^cE + d^cE [\Lambda, d^cE] = [\Lambda, (d^cE)^2]$, whereas from (1.6.7) we get $(d^cE)^2 \Phi = -J(R^\nabla \wedge J^{-1} \Phi) = -(JR^\nabla) \wedge \Phi$: similarly, $\delta^cE d^\nabla + d^\nabla \delta^cE = -[\Lambda, d^\nabla]d^\nabla - d^\nabla [\Lambda, d^\nabla] = -[\Lambda, (d^\nabla)^2]$, whereas $(d^\nabla)^2 \Phi = -R^\nabla \wedge \Lambda \Phi$.

1.15. Laplace and twisted Laplace operators

For any riemannian manifold $(M, g)$, the riemannian Laplace operator, or simply laplacian, $\Delta$, acting on exterior forms, is defined by $\Delta = \delta d + d\delta$.

For any $g$-compatible almost-complex structure $J$, we define the twisted Laplace operator $\Delta^c$ by $\Delta^c = \delta^c d^c + d^c \delta^c = J\Delta J^{-1}$.

In the case when $J$ is integrable, we also define the Dolbeault Laplace operators or Dolbeault laplacians, $\Box$ and $\overline{\Box}$, defined by

\[ \Box := \partial^* \partial + \partial \partial^*, \quad \overline{\Box} := \overline{\partial}^* \overline{\partial} + \overline{\partial} \overline{\partial}^*, \]

where the operators $\partial$, $\bar{\partial}$ and their formal adjoints $\partial^*$, $\bar{\partial}^*$ with respect to the hermitian global inner product have been introduced in Remark 1.11.3. Clearly, both $\Box$ and $\overline{\Box}$ preserve the type of exterior forms, cf. Section 1.9. We then have

Proposition 1.15.1. On any compact Kähler manifold, we have

\[ \Delta = \Delta^c = 2\overline{\Box} = 2\Box. \]

In particular, $\Delta$ preserves the type of exterior forms, meaning that

\[ (\Delta \psi)^{r,s} = \Delta (\psi^{r,s}), \]

for any $p$-form $\psi$ and any $r, s$ with $r + s = p$.

Proof. By using the Kähler identities (1.14.1), combined with (1.11.9), we get

\[
\begin{align*}
\Delta^c &= d^c \delta^c + \delta^c d^c \\
&= d^c [d, \Lambda] + [d, \Lambda] d^c \\
&= d^c d\Lambda - d^c d\Lambda + d\Lambda d^c - d\Lambda d^c \\
&= -d^c d\Lambda + d\Lambda d^c + \Lambda d^c d \\
&= d(\Lambda d^c - d^c \Lambda) + (\Lambda d^c - d^c \Lambda)d, \\
&= d\delta + \delta d = \Delta,
\end{align*}
\]

use $\delta^c = [d, \Lambda]$, use $[\Lambda, d^c] = \delta$, etc.
whereas, by (1.11.14)–(1.11.17), we get
\[
\square = \frac{1}{4}(d + id^c)(\delta - i\delta^c)(d + id^c)
\]
\[
= \frac{1}{4}(d\delta + \delta d + d^c\delta^c + \delta^c d^c)
\]
\[
+ \frac{i}{4}(-\delta^c d - d\delta^c + d^c\delta + \delta d^c)
\]
\[
= \frac{1}{4}(d\delta + \delta d + d^c\delta^c + \delta^c d^c)
\]
\[
= \frac{1}{4}(\Delta + \Delta^c) = \frac{1}{2}\Delta = \frac{1}{2}\Delta^c,
\]
and similarly for \(\Box\).

From Proposition 1.15.1, we readily infer the following well-known topological obstructions to the existence of Kähler structures in the compact case:

**Proposition 1.15.2.** The odd order Betti numbers, \(b_{2k+1}\), of any compact Kähler manifold are even, whereas the even order Betti numbers, \(b_{2k}\), are all positive.

**Proof.** For any compact riemannian manifold \((M, g)\) the classical Hodge theory asserts that the (complex) dimension of the kernel of \(\Delta\), acting on the space of complex \(p\)-form, i.e. the space, \(\mathcal{H}_g^p\), of complex \(g\)-harmonic \(p\)-forms, is finite and a topological invariant, equal to the dimension, \(b_p\), of the cohomology space \(\mathcal{H}_g^p\), called the \(p\)-th Betti number, cf. e.g. [193]. Because of (1.15.3), when \(g\) is Kähler, \(\mathcal{H}_g^p\) splits as a direct sum \(\mathcal{H}_g^p = \bigoplus_{r+s=p} \mathcal{H}_g^{r,s}\), where \(\mathcal{H}_g^{r,s}\) denotes the space of harmonic forms of type \((r, s)\), for all pairs of non negative integers \(r, s\) such that \(r + s = p\); we then have \(b_p = \sum_{r+s=p} h^{r,s}\), where the \(h^{r,s}\) — the Hodge numbers of \((M, J)\) — denote the (complex) dimension of \(\mathcal{H}_g^{r,s}\). Since \(\Delta\) is real, the conjugation induces a \(\mathbb{C}\)-antilinear isomorphism from \(\mathcal{H}_g^{r,s}\) to \(\mathcal{H}_g^{s,r}\) for any such pair \(r, s\), so that \(h^{r,s} = h^{s,r}\). If \(p = 2k + 1\) is odd, we then get \(b_p = 2\sum_{r=0}^{k} h^{2k+1-r}\) is even. The last assertion of Proposition 1.15.2 follows from the fact that, for each \(k = 1, \ldots, m\), \(\Omega^k := [\omega^k]\) is not zero, as \(\Omega^m \neq 0\) (recall that \(\omega^m / m!\) is the volume form of \(g\), hence does not integrate to zero). \(\Box\)

**Proposition 1.15.3.** For any Kähler manifold, we have the following identities:

\[
[\Delta, L] = 0, \quad [\Delta, \Lambda] = 0.
\]

In particular, for any harmonic exterior form \(\psi\), \(L\psi = \omega \wedge \psi\) and \(\Lambda\psi\) are harmonic as well.

**Proof.** Since \([\Delta, \Lambda]^* = -[\Delta, L]\), the second identity in (1.15.4) follows from the first one. Since the Kähler form \(\omega\) is closed, we have \([d, L] = 0\). By using the second identity in (1.14.2), we then get
\[
[\Delta, L] = -(dd^c + d^c d),
\]
on any almost-Kähler manifold. In the Kähler case, we have \(dd^c + d^c d = 0\) by Proposition 1.11.2, hence \([\Delta, L] = 0\). The last assertion readily follows from (1.15.4). \(\Box\)
Remark 1.15.1. Proposition 1.15.2 provides an easy non-existence criterion of Kähler metrics on a given compact complex manifold \((M,J)\). For example, it directly follows from Proposition 1.15.2 that the so-called Calabi-Eckmann complex manifolds, whose underlying spaces are products of odd-dimensional spheres \(S^{2p+1} \times S^{2q+1}\), with \(p \geq 0\) and \(q > 0\), admit no Kähler metric, cf. [48].

Remark 1.15.2. For any (real) function \(f\) the Kähler identities (1.14.1) combined with the general identity
\[
\Lambda (dd^c + d^c d) f = 0,
\]
which readily follows from (1.11.10), imply
\[
\Delta f = \Delta^c f = -\Lambda (df^c f) = \Lambda (d^c df),
\]
on any symplectic almost-hermitian manifold.

1.16. The Akizuki-Nakano identity

For any vector bundle \(E\) over \(M\), equipped with a fiberwise inner product \(h\) and an \(h\)-compatible linear connection \(\nabla\), we can define a riemannian Laplace operator \(\Delta^\nabla := \delta^\nabla d^\nabla + d^\nabla \delta^\nabla\) and a twisted Laplace operator
\[
\Delta^c,^\nabla := \delta^c,^\nabla d^c,^\nabla + d^c,^\nabla \delta^c,^\nabla = J \Delta^\nabla J^{-1}.
\]
If, moreover, \(E = (E,h)\) is a hermitian holomorphic vector bundle over a complex manifold \((M,J)\), and \(\nabla\) is the corresponding Chern connection, we can similarly define the Dolbeault Laplace operator by \(\square^E = (\bar{\partial}^E)^* \bar{\partial}^E + \bar{\partial}^E (\bar{\partial}^E)^*\), as well as \(\square^\nabla = (\partial^\nabla)^* \partial^\nabla + \partial^\nabla (\partial^\nabla)^*\), where the operators \(\bar{\partial}^E\) and \(\partial^\nabla\), acting on \(E\)-valued exterior forms of any type were defined by (1.11.26)–(1.11.27) in Remark 1.11.4 of Section 1.6. If, moreover, the almost hermitian structure \((g,J,\omega)\) is Kähler, we have:

Proposition 1.16.1. Let \((E,h)\) be a hermitian holomorphic vector bundle over a Kähler manifold \((M,g,J,\omega)\). Then,
\[
\Delta^\nabla = \Delta^c,^\nabla = \square^\nabla + \square^E,
\]
and
\[
\square^E = \frac{1}{2} \left( \Delta^\nabla - i [\Lambda, R^\nabla] \right), \quad \square^\nabla = \frac{1}{2} \left( \Delta^\nabla + i [\Lambda, R^\nabla] \right),
\]
where \(R^\nabla\) denotes the curvature of the Chern connection \(\nabla\) of \((E,h)\). In particular:
\[
\square^E - \square^\nabla = i [\Lambda, R^\nabla],
\]
with
\[
[\Lambda, R^\nabla] \Phi = \Lambda (R^\nabla \wedge \Phi) - R^\nabla \wedge \Lambda \Phi,
\]
for any \(E\)-valued exterior form \(\Phi\).

Proof. The proof of (1.16.3)–(1.16.4) is quite similar to the proof of Proposition 1.15.1, by using: (i) Proposition 1.11.3, which guarantees that \(d^\nabla d^c,^\nabla + d^c,^\nabla d^\nabla = 0\), as \(R^\nabla\) is \(J\)-invariant; (ii) the extended Kähler identities of Proposition 1.14.3, which are formally the same as in the scalar case; (iii) the identity (1.14.6), which replaces the identity (1.14.3) used in the scalar
1.17. THE $\mathbb{d\bar{d}}$-LEMMA

case. Notice that $i[R^\nabla, \Lambda]$ is a hermitian endomorphism of $E \otimes \Lambda^\bullet M$. We thus have

$$\Delta^{c,\nabla} = d^{c,\nabla} \delta^{c,\nabla} + \delta^{c,\nabla} d^{c,\nabla}$$

use $\delta^{c,\nabla} = [d^{\nabla}, \Lambda]$

$$= d^{c,\nabla} d^{\nabla} \Lambda - d^{c,\nabla} \Lambda d^{\nabla} + d^{\nabla} \Lambda d^{c,\nabla} - \Lambda d^{\nabla} d^{c,\nabla}$$

use $\delta^{c,\nabla} = [d^{\nabla}, \Lambda]$

$$= -d^{\nabla} d^{c,\nabla} \Lambda - d^{c,\nabla} \Lambda d^{\nabla} + d^{\nabla} \Lambda d^{c,\nabla} + \Lambda d^{\nabla} d^{c,\nabla}$$

$$= d^{\nabla} (\Lambda d^{c,\nabla} - d^{c,\nabla} \Lambda) + (\Lambda d^{c,\nabla} - d^{c,\nabla} \Lambda) d^{\nabla},$$

use $[d^{\nabla}, \Lambda] = \delta^{\nabla}$, and

and

$$\Box^{\nabla} = \frac{1}{4} (d^{\nabla} + id^{c,\nabla}) (\delta^{\nabla} - i\delta^{c,\nabla}) + (\delta^{\nabla} - i\delta^{c,\nabla}) (d^{\nabla} + id^{c,\nabla})$$

$$= \frac{1}{4} (d^{\nabla} \delta^{\nabla} + \delta^{\nabla} d^{\nabla} + d^{c,\nabla} \delta^{c,\nabla} + \delta^{c,\nabla} d^{c,\nabla})$$

use (1.14.6) together with $JR^{\nabla} = R^{\nabla}$

$$= \frac{1}{2} (\Delta^{\nabla} + \Delta^{c,\nabla})$$

use (1.16.3) known as the Akizuki–Nakano identity, cf. [2], [66].

Similarly

$$\Box^{E} = \frac{1}{4} (d^{\nabla} - id^{c,\nabla}) (\delta^{\nabla} + i\delta^{c,\nabla}) + (\delta^{\nabla} + i\delta^{c,\nabla}) (d^{\nabla} - id^{c,\nabla})$$

$$= \frac{1}{4} (d^{\nabla} \delta^{\nabla} + \delta^{\nabla} d^{\nabla} + d^{c,\nabla} \delta^{c,\nabla} + \delta^{c,\nabla} d^{c,\nabla})$$

use (1.14.6) together with $JR^{\nabla} = R^{\nabla}$

$$= \frac{1}{2} \Delta^{\nabla} + \frac{i}{2} [\Lambda, R^{\nabla}].$$

1.17. The $\mathbb{d\bar{d}}$-lemma

For any $p$-form $\psi$ defined on an oriented, compact riemannian manifold, the Hodge decomposition of $\psi$ is written as follows

\begin{equation}
\psi = \psi_H + \Delta G \psi = \psi_H + d\delta G \psi + \delta d G \psi,
\end{equation}

where $\psi_H$ is harmonic and $G$ is the Green operator for the laplacian $\Delta$. Since, $\Delta = \Delta^c$, the corresponding Green operators coincide as well and we also have

\begin{equation}
\psi = \psi_H + \Delta^c G \psi = \psi_H + d^c \delta^c G \psi + \delta^c d^c G \psi.
\end{equation}

The $\mathbb{d\bar{d}}$-lemma then reads (see [64])
LEMMA 1.17.1. Let \((M, g, J, \omega)\) be a compact Kähler manifold. Then the Hodge decomposition of any real \(dd^c\)-closed p-form \(\psi\) can be written as

\[
\psi = \psi_H + d\delta G\psi - d\delta dG^2\psi
\]

where \(\psi_H\) is harmonic.

Proof. From (1.17.1) and from \(dd^c\psi = 0\), we infer \(0 = dd^c\delta dG\psi = -d\delta dG\psi\), so that \(\delta dG\psi = -\delta dd^c\psi = 0\). It follows that \(dd^c\psi = 0\). The harmonic part of \(dG\psi\) with respect to \(\Delta\), hence also with respect to \(\Delta^c = \Delta\), equals 0. From (1.17.2), we then infer \(dG\psi = d\delta GdG\psi\), so that \(\delta dG\psi = -d\delta d^cGdG\psi = -d^c\delta d^cG^2\psi\). (In this proof, we have used the identities \(\delta d^c = -d^c\delta\) and \(dd^c = -d^c d\), as well as \(dG = Gd\)).

As a direct corollary, we get

PROPOSITION 1.17.1. (Same hypotheses as in Lemma 1.17.1). Let \(\psi\) be any \(J\)-invariant p-form satisfying \(\psi = d\phi\), for some \((p-1)\)-form \(\phi\). Then,

\[
\psi = dd^c\tilde\psi,
\]

for some \((p-2)\)-form \(\tilde\psi\).

Proof. Since \(J\psi = \psi\), \(\psi\) is \(d^c\)-closed and \(\phi\) is then \(dd^c\)-closed. It then follows from Lemma 1.17.1 that \(\phi = \phi_H + d\delta G\phi = d\delta dG^2\phi\), so that \(\psi = d\phi = dd^c\tilde\psi\), with \(\tilde\psi = -d\delta dG^2\phi = -d\delta G^2\psi\).

The above proposition is most often used when \(p = 2\) and is then stated as follows:

PROPOSITION 1.17.2. Let \(\psi_0, \psi_1\) be any two real \(J\)-invariant closed 2-forms defined on a compact Kähler manifold \((M, g, J, \omega)\) and suppose that \(\psi_0, \psi_1\) determine the same de Rham class in \(H^2_{dR}(M, \mathbb{R})\). Then there exists a real function \(f\), uniquely defined up to an additive constant, such that

\[
\psi_1 - \psi_0 = dd^c f.
\]

Proof. (1.17.5) is a direct application of Proposition 1.17.1 for \(\psi = \psi_1 - \psi_0\). If \(dd^c f = 0\), then, by (1.15.6), \(\Delta f = 0\), i.e. \(f\) is harmonic, hence constant, as \(M\) is compact.

REMARK 1.17.1. In general, for any complex manifold \((M, J)\), the kernel of the operator \(dd^c\) acting on \(C^\infty(M, \mathbb{R})\) is the space of those real smooth functions \(f\) on \(M\) which are locally the real part of a holomorphic function. Indeed, if \(dd^c f = 0\), then, by the Poincaré Lemma, \(d^c f = \text{loc. dh}\), for some local (real) function \(h\) and \(F := f + ih\) is then holomorphic, as \(\partial F = \frac{1}{2}(d - i\partial d^c)(f + ih) = \frac{1}{2}(df + d^c h) + \frac{1}{2}(dh - dF) = 0\). If \((M, J)\) is compact, it is true, even in the case that \((M, J)\) is not of kählerian type, that the kernel of \(dd^c\) acting on \(C^\infty(M, \mathbb{R})\) is reduced to the space of constant functions. This fact cannot be directly deduced from the above local statement, as there is no guarantee a priori that an element \(f\) of the kernel of \(dd^c\) be the real part of a globally defined holomorphic function (which would then be constant, as \(M\) is compact). On the other hand, the operator \(f \mapsto \tilde\Delta := -\Delta dd^c f\) is defined for any almost-hermitian metric \(g\) on \((M, J)\); except if the structure is almost-Kähler, \(\tilde\Delta\) is different from the riemannian Laplace operator \(\Delta_g\); it nevertheless shares many properties of \(\Delta_g\): it is as an elliptic, second
order linear differential operator acting on $C^\infty(M, \mathbb{R})$, such that $\tilde{\Delta}(1) = 0$; this is sufficient to conclude that the kernel of $\tilde{\Delta}$ is reduced to the space of constants, as $M$ is compact, cf. [103].

A related result is the following

**Proposition 1.17.3.** Any $J$-invariant real closed 2-form $\psi$ on a Kähler manifold $(M, g, J, \omega)$ can be locally written as

$$\psi = \text{loc} \, dd^c f,$$

for some local real function. For any two such functions $f, f'$ defined on a same open subset of $M$, the difference $f' - f$ is locally the real part of a holomorphic function.

**Proof.** In contrast to the previous proposition, Proposition 1.17.3 is of local character and does not require that $M$ be compact. Its proof readily follows from the Poincaré lemma, which asserts that any closed form is locally exact, and from the Dolbeault lemma, which similarly asserts that any $\bar{\partial}$-closed complex form $\phi$ of type $(p, q)$, $q > 0$, is locally of the form $\phi = \bar{\partial} \phi$, for some $(p, q - 1)$-form $\phi$, cf. e.g. [156], [93]. The argument then goes as follows. Since $\psi$ is closed, by the Poincaré lemma it can locally be written as $\psi = \text{loc} \, d\alpha$, for some real 1-form $\alpha$. Since $\psi$ is $J$-invariant, the $(0, 1)$-part $\alpha^{0,1}$ of $\alpha$ is $\bar{\partial}$-closed; by the Dolbeault lemma there then exists a complex function $F$, such that $\alpha = \text{loc} \, \bar{\partial} F = \frac{1}{2} (d - i d^c) F$, possibly, on a smaller open set. We then get $\psi = \text{loc} \, \frac{1}{2} d((d - i d^c) F + (d + i d^c) F) = dd^c f$, with $f = \text{Im}(F)$. This proves the first part of the proposition. The second part has been proved in Remark 1.17.1. □

**Definition 1.** For any Kähler manifold $(M, g, J, \omega)$, a (local) **Kähler potential** is a real function $\phi$ defined on some open subset $U$ of $M$, such that

$$\omega = \text{loc} \, dd^c \phi.$$

By Proposition 1.17.3, such a potential exists around any point of $M$, and is then defined up the addition of a function which is locally the real part of a holomorphic function. It may happen that $(M, g, J, \omega)$ admits a globally defined Kähler potential: for example, denote by $(g_0, J_0, \omega_0)$ the **standard flat Kähler structure** of the standard hermitian vector space $\mathbb{C}^m$ (viewed as a $2m$-dimensional real manifold), defined by

$$g_0(X, Y) = \Re \langle X, Y \rangle, \quad J_0 X = i X, \quad \omega_0(X, Y) = -\text{Im} \langle X, Y \rangle,$$

for any $u$ in $\mathbb{C}^m$ and any two $X, Y$ in $T_u \mathbb{C}^m \cong \mathbb{C}^m$, where $\langle \cdot, \cdot \rangle$ here stands for the standard hermitian inner product. It is then an easy exercise to check that

$$\omega_0 = \frac{1}{4} dd^c r^2,$$

where $r^2$ is the square norm function determined by $\langle \cdot, \cdot \rangle$, meaning that $\frac{1}{4} r^2$ is a global Kähler potential. Note that the standard flat Kähler structure is invariant under the natural (linear) action of the unitary group $U(m)$, and, up to an additive constant, $\frac{1}{4} r^2$ is the unique $U(m)$-invariant (global) Kähler potential.
On the other hand, on a compact Kähler manifold \((M, g, J, \omega)\), no globally defined Kähler potential can possibly exist, as the de Rham class \(\Omega = [\omega]\) is then non-zero, see Proposition 1.15.2.

### 1.18. Riemannian curvature and Bianchi identities

For a general riemannian manifold, the **riemannian curvature** \(R\) is defined as the curvature of the Levi-Civita connection, hence by:

\[
R_{X,Y}Z = D_{[X,Y]}Z - [D_X, D_Y]Z,
\]

for all vector fields \(X, Y, Z\) (see (1.6.6 and Remark 1.6.1).

The riemannian curvature satisfies the **algebraic Bianchi identity** — or **first Bianchi identity** — and the **differential Bianchi identity** — or **second Bianchi identity** — respectively defined by

\[
\text{bianchi1}\quad R_{X_1,X_2,X_3} + R_{X_2,X_3,X_1} + R_{X_3,X_1,X_2} = 0,
\]

and

\[
\text{bianchi2}\quad (D_{X_1}R)_{X_2,X_3} + (D_{X_2}R)_{X_3,X_1} + (D_{X_3}R)_{X_1,X_2} = 0,
\]

for any vector fields \(X_1, X_2, X_3\).

The differential Bianchi identity (1.18.3) is actually a special case of a very general identity, which holds for any linear connection \(\nabla\) on any vector bundle \(E\) over some manifold \(M\), and which can be written as follows

\[
\text{bianchi2gen}\quad d\nabla R\nabla = 0,
\]

where, we recall, the curvature \(R\nabla\) of \(\nabla\) is viewed as a \(\text{End}(E)\)-valued 2-form on \(M\) and we still denote \(\nabla\) the induced covariant derivative on \(\text{End}(E)\). Then, (1.18.4) is a direct consequence of (1.6.7) and of the associativity relation \(d\nabla \circ (d\nabla \circ d\nabla) = (d\nabla \circ d\nabla) \circ d\nabla\).

It is then as simple exercise to show that (1.18.4) reduces to (1.18.3) when \(\nabla\) is the Levi-Civita connection of a riemannian manifold.

The algebraic Bianchi identity (1.18.2) also holds in a larger setting, namely for any torsion-free linear connection defined on \(TM\), and is again deduced from (1.6.7) in the following way. Let \(\nabla\) be any linear connection on \(TM\) and denote by \(\text{T}\nabla\) its torsion, defined by \(\text{T}\nabla_{X,Y} = \nabla_X Y - \nabla_Y X - [X,Y]\); the torsion can then be viewed as a \(TM\)-valued 2-form and, by comparing with (1.6.3), we infer that it can be written as \(\text{T}\nabla = d\nabla I\), where \(I\) stands for the identity operartor of \(TM\), viewed as \(TM\)-valued 1-form; then (1.6.7) reads as

\[
\text{bianchi1gen}\quad R\nabla \wedge I = -d\nabla T\nabla,
\]

where the lhs is exactly the rhs of (1.18.2) when \(R\) is replaced by \(R\nabla\); if \(T\nabla \equiv 0\), i.e. if \(\nabla\) is torsion-free, we thus get (1.18.2); in the general case, (1.18.5) can be regarded as a **generalized algebraic Bianchi identity**, which holds for the curvature of any linear connections on \(TM\).

In the riemannian case, the algebraic Bianchi identity (1.18.2) implies the following **weak** (algebraic) **Bianchi identity**:

\[
\text{weakbianchi}\quad (R_{X_1,X_2,X_3}, X_4) = (R_{X_3,X_4,X_1}, X_2).
\]
The vector bundle $\Lambda^2 M := \Lambda^2 (T^* M)$ of 2-forms, the vector bundle $\Lambda^2 (TM)$ of 2-vectors and the vector bundle $A(M)$ of skew-symmetric endomorphisms of $TM$ — or $T^* M$ — are all naturally identified with each other via the riemannian duality induced by $g$: the riemannian curvature $R$ can then be viewed in a tautological way as an operator acting on any of them, by just setting $(R(X_1 \wedge X_2), X_3 \wedge X_4) := (R_{X_1, X_2} X_3, X_4)$. In this rôle, $R$ will be called the curvature operator and (1.18.6) simply says that the curvature operator $R$ is symmetric.

The Ricci tensor, $r$, of a riemannian manifold is defined by:

\begin{equation}
\text{(1.18.7)}
r(X, Y) = \text{tr} \{ Z \to R_{X, Z} Y \} = \sum_{j=1}^{n} (R_{X, e_j} Y, e_j),
\end{equation}

for any auxilliary orthonormal basis $\{e_1, \ldots, e_n\}$.

Again, the weak algebraic Bianchi identity (1.18.6) readily implies that the Ricci tensor $r$ is symmetric.

The scalar curvature $s$ is defined as the trace of $r$ with respect to $g$, namely by

\begin{equation}
\text{(1.18.8)}
s = (r, g).
\end{equation}

Notice that the derivation of $s$ from $r$ does require the use of the metric $g$, while the derivation of $r$ from $R$ does not.

From the differential Bianchi identity (1.18.3) we easily deduce the contracted Bianchi identity:

\begin{equation}
\text{(1.18.9)}
\delta r + \frac{1}{2} ds = 0
\end{equation}

which relates the codifferential of the Ricci tensor — see Section 1.10 — to the differential of the scalar curvature of any riemannian metric.

A riemannian metric $g$ of dimension $n > 2$ is called an Einstein metric if its Ricci tensor $r$ is proportional to $g$: we then have

\begin{equation}
\text{(1.18.10)}
r = \frac{s}{n} g.
\end{equation}

From (1.18.9) we then infer that the scalar curvature $s$ is constant: indeed, if $r = \frac{s}{n} g$, (1.18.9) simply reads $(\frac{1}{2} - \frac{1}{n}) ds = 0$ which implies $ds = 0$, as $n \neq 2$. If $n = 2$, (1.18.10) holds for any riemannian metric and we say that $g$ is an Einstein metric if $s$ is constant: $(M, g)$ is then of constant sectional curvature — or Gauss curvature — equal to $\frac{s}{2}$.

1.19. The Ricci form of a Kähler structure

In the Kähler context of these notes, the riemannian curvature operator $R$ — cf. Section 1.18 — acts trivially on the $J$-anti-invariant part of $\Lambda^2 M$, hence appears as a (symmetric) endomorphism of the $J$-invariant part $\Lambda^{1,1} M$.

Alternatively, the riemannian curvature $R$ of a Kähler manifold can be viewed as a $J$-invariant 2-form with values in $\Lambda^{1,1} M$.

This is consistent with the fact that the Levi-Civita connection $D$ coincides with the canonical Chern connection of $(TM, J)$ (cf. Proposition
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1.8.1), so that \( R \) coincides with the curvature of the canonical Chern connection of \((TM, J)\), when the latter is regarded as a hermitian holomorphic vector bundle, cf. Section 1.6.

The Ricci tensor \( r \) is then \( J \)-invariant, i.e. satisfies \( r(JX, JY) = r(X, Y) \), and the Ricci form is defined as the associated \( J \)-invariant 2-form, denoted by \( \rho \), defined by:

\[
\rho(X, Y) = r(JX, Y).
\]

Notice that the Ricci form \( \rho \) is associated to the Ricci tensor in the same way as the Kähler form \( \omega \) is associated to the riemannian metric \( g \). Also notice that in terms of the Ricci form, the riemannian scalar curvature \( s \) is expressed by

\[
s = 2 \Lambda \rho.
\]

**Lemma 1.19.1.** The Ricci form \( \rho \) has the following alternative description:

\[
\rho = \frac{1}{2} \sum_{j=1}^{n} (R_{e_j, Je_j} X, Y) = R(\omega),
\]

for any auxiliary orthonormal frame \( \{e_1, \ldots, e_n\} \).

**Proof.** By using the algebraic Bianchi identity (1.18.2) and the very definition (1.18.7), the second expression in (1.19.3) can be rewritten as:

\[
\frac{1}{2} \sum_{j=1}^{n} (R_{e_j, Je_j} X, Y) = -\frac{1}{2} \sum_{j=1}^{n} ((R_{Je_j, e_j} X, Y) + (R_{X, e_j} Je_j, Y))
\]

\[
= \frac{1}{2} \sum_{j=1}^{n} ((R_{e_j, JX e_j} Y) + (R_{X, e_j} e_j, JY))
\]

\[
= \frac{1}{2} r(JX, Y) - \frac{1}{2} r(X, JY) = r(JX, Y) = \rho(X, Y).
\]

We then get the third expression by observing that the Kähler form \( \omega \) is also \( \omega = \frac{1}{2} \sum_{j=1}^{n} e_j \wedge Je_j \), for any orthonormal basis \( \{e_1, \ldots, e_n\} \), viewed as basis of covectors via the riemannian duality (compare with the more complicated computation (9.4.9) in the almost Kähler case). \( \square \)

In (1.19.3), \( R(\omega) \) denotes the image of the Kähler form \( \omega \) by the riemannian curvature operator \( R \). Note that due to the symmetry of the curvature operator, \( R(\omega) \) is the same as the real 2-form obtained by taking the trace of \(-J \circ R\), when the latter is viewed as a hermitian operator valued 2-form, i.e.

\[
R(\omega)(X, Y) = \text{tr}(-J \circ R_{X, Y}),
\]

where the trace is here understood as the trace of a \( \mathbb{C} \)-linear hermitian operator (see Section 1.12).

This has the following interpretation. Consider the anti-canonical complex line bundle \( K^{-1}_M := \Lambda^m((TM, J)) \), where \( \Lambda^m((TM, J)) \) denotes the top (complex) exterior power of \((TM, J)\). This is actually a hermitian holomorphic line bundle, as \((TM, J)\) is itself a hermitian holomorphic vector bundle (see Sections 1.8, 1.6 and 1.7) and the corresponding Chern connection is
then the connection induced by the Chern connection of $(TM, J)$; it follows that the Chern curvature of $K^{-1}_M$, say $R_{X,Y}^{K^{-1}_M}$, is equal to the (complex) trace of the Chern curvature of $(TM, J)$, which, by Proposition 1.8.1, is the Riemannian curvature $R$; we then have $R_{X,Y}^{K^{-1}_M} = \text{tr}(R_{X,Y})$; here, $\text{tr}(R_{X,Y})$ is understood as the (complex) trace of $R_{X,Y}$ as an anti-hermitian, C-linear operator acting on $(TM, J) \cong T^1,0M$; $R_{X,Y}^{K^{-1}_M}$ is then a purely imaginary 2-form and can be alternatively written as $R_{X,Y}^{K^{-1}_M} = i \text{tr}(-J \circ R_{X,Y})$, where $-J \circ R_{X,Y}$ is hermitian, so that $\text{tr}(-J \circ R_{X,Y})$ is real; by comparing with (1.19.4) and (1.19.3), we eventually end up with

$$R_{X,Y}^{K^{-1}_M} = i \rho.$$  

We have thus established the following basic series of facts concerning the Ricci form:

**Proposition 1.19.1.** For any Kähler manifold $(M,g,J,\omega)$, the Ricci form $\rho$ is the curvature form of the Chern connection of the anti-canonical line bundle $K^{-1}_M$ relatively to the induced hermitian inner product. In particular $\rho$ is closed and its de Rham class, $[\rho]$, in $H^2(M,\mathbb{R})$ is equal to $2\pi c_1(M,J)$, where $c_1(M,J)$ is defined as the real Chern class of $K^{-1}_M$, which is also the first (real) Chern class of $(TM, J)$. The de Rham class $[\rho]$ of $\rho$ is then the same for all Kähler metrics on $(M,J)$.

Moreover, $\rho$ has the following local expression:

$$\rho = -\frac{1}{2} \dd c \log \frac{v_g}{v_0},$$

where $v_0$ stands for the volume form of the flat Kähler metric determined by any choice of a (local) holomorphic coordinate system, and, for any two Riemannian metrics, $g_0$ and $g$ — both Kähler with respect to $J$ — the corresponding Ricci forms $\rho_0$ and $\rho$ are related by

$$\rho - \rho_0 = -\frac{1}{2} \dd c \log \frac{v_g}{v_{g_0}}.$$

**Proof.** The first statement follows from the above discussion and from Remark 1.7.1. The local expression (1.19.6) follows from Proposition 1.7.1 in the following way. For any point $x$ of $M$, let $z_1, \ldots, z_m$ be any holomorphic coordinate system defined on some open neighbourhood $U$ of $x$ in $M$; the Kähler form of the corresponding flat Kähler structure is then $\frac{i}{2} \sum_{r=1}^{m} dz_r \wedge d\bar{z}_r$; the corresponding volume form is $v_0 := \prod_{r=1}^{m} \frac{1}{2} dz_r \wedge d\bar{z}_r$, whereas from (1.2.2) we easily infer

$$v_g = 2^m |\partial/\partial z_1 \wedge \ldots \wedge \partial/\partial z_m|^2 v_0.$$ 

In (1.19.8), $\partial/\partial z_1 \wedge \ldots \wedge \partial/\partial z_m$ is a (nowhere vanishing) holomorphic section of $K^*(M)$ on $U$ and $|\partial/\partial z_1 \wedge \ldots \wedge \partial/\partial z_m|^2$ stands for its square norm with respect to the induced hermitian inner product. Then, (1.19.6) readily follows from (1.19.8) and from the general formula (1.7.2), whereas (1.19.7) — where the rhs is now globally defined — is an immediate consequence of (1.19.6).
Remark 1.19.1. From (1.19.6) we deduce that the Ricci form $\rho$ of any Kähler structure $(g, J, \omega)$ is entirely determined by the complex structure $J$ and the volume form $v_g$ of $g$.

Proposition 1.19.2. The Ricci form $\rho$ of a Kähler manifold $(M, g, J, \omega)$ is co-closed if and only if the scalar curvature $s$ of $g$ is constant. If $M$ is compact, $\rho$ is then harmonic if and only if $s$ is constant. If $M$ is compact, of complex dimension $m > 1$, and if the first Chern class $c_1(M, J)$ is related to the Kähler class $\Omega := [\omega]$ by $c_1(M, J) = \frac{k}{2\pi} \Omega$, for some real number $k$, then $g$ is an Einstein metric if and only if $s$ is constant; $s$ is then equal to $\frac{2km}{m}$.

Proof. In the Kähler context, the contracted Bianchi identity (1.18.9) can be rewritten as

$$\delta \rho = -\frac{1}{2} d^c s$$

(easy verification); this proves the first assertion. For any compact Riemannian manifold, an exterior form $\phi$ is harmonic — meaning that $\Delta \phi = 0$ — if and only if $\phi$ is closed and co-closed (direct consequence of $\Delta = \delta d + d \delta$, as $\delta$ is the (formal) adjoint of $d$); this proves the second assertion. If $c_1(M, J) = \frac{k}{2\pi} \Omega$ and $\rho$ is harmonic, $\rho$ and $k\omega$ are both harmonic representatives of $2\pi c_1(M, J)$, so that $\rho = k\omega$; equivalently, $r = kg$, meaning that $g$ is an Einstein metric and $k = \frac{s}{2m}$.

For a general (connected) compact Kähler manifold, the defect for the Ricci form to be harmonic is measured by the Ricci potential, $p_g$, defined — via the $dd^c$-lemma 1.17.1 — by

$$\rho = \rho_H - \frac{1}{2} dd^c p_g,$$

where $\rho_H$ denotes the harmonic part of the Ricci form $\rho$ with respect to $g$, and by

$$\int_M p_g v_g = 0.$$

Since the trace of any harmonic, $J$-invariant 2-form is constant — easy consequence of the Kähler identity (1.14.1) — we infer that the Ricci potential and the scalar curvature are related by

$$s_g - \bar{s} = \Delta p_g,$$

where $\bar{s} = \frac{1}{V} \int_M s_g v_g$ denotes the mean value of $s_g$ with respect to $v_g$. In terms of the Green operator $G$ introduced in Section 1.17, we then have

$$p_g = G(s_g - \bar{s}).$$

By its very definition, the Ricci potential is identically zero if and only if the Ricci form is harmonic; by (1.19.12)-(1.19.13), we retrieve the fact already observed in Proposition 1.19.2 that this happens if and only if $s_g$ is constant.

Remark 1.19.2. As explained in Appendix A, the curvature of a Kähler metric splits into three components: a scalar component, identified with the scalar curvature, a second component, identified with the trace-free part of the Ricci tensor, and the Bochner tensor. A Kähler metric whose Bochner tensor is zero is called Bochner-flat, or Bochner-Kähler. The holomorphic
sectional curvature $H$ of a Kähler metric is the restriction of the sectional curvature to orthonormal pairs of the form $X, JX$; for any unit vector $X$, we then have $H(X) = g(R_{X, JX} X, JX)$. The holomorphic sectional curvature entirely determines the whole curvature and a Kähler metric is of constant holomorphic sectional curvature if and only if it is both Kähler-Einstein and Bochner-flat, i.e. its curvature is reduced to its scalar component. The curvature is then of the form

$$R_{X, Y} = rac{c}{4} (X \wedge Y + JX \wedge JY + 2\omega(X, Y) J),$$

for any vector fields $X, Y$, where $c$ denotes the (constant) value of the holomorphic sectional curvature (the rhs has to be regarded as a skew-symmetric, $J$-linear endomorphism via the metric $g$). The (constant) scalar curvature $s$ is then given by $s = m(m+1)c$.

### 1.20. The Calabi-Yau theorem

For any de Rham class $\Omega$ in $H^2_{dR}(M, \mathbb{R})$, we denote by $\mathcal{M}_\Omega$ the set of Kähler metrics on $(M, g, J, \omega)$ whose Kähler form $\omega$ belongs to $\Omega$. If $\mathcal{M}_\Omega$ is non-empty, $\Omega$ is called a Kähler class and $\mathcal{M}_\Omega$ is then an infinite-dimensional Fréchet manifold. More precisely, fix any $g_0$ in $\mathcal{M}_\Omega$, with Kähler class $\omega_0$. Then, by Proposition 1.17.2, the Kähler class, $\omega$, of any $g$ in $\mathcal{M}_\Omega$ can be written as $\omega = \omega_0 + dd^c \phi$ for some real $C^\infty$ function, uniquely defined up to an additive constant. We then obtain a natural identification of $\mathcal{M}_\Omega$ with the space of real $C^\infty$ functions $\phi$ normalized, say, by $\int_M \phi v_{g_0} = 0$ — see however the end of Section 4.1 for a more natural normalization — and such that $\omega = \omega_0 + dd^c \phi$ is positive, meaning that $g := \omega(\cdot, J \cdot)$ is a riemannian metric. Recall — cf. Remark 1.7.1 — that for any Kähler structure $(g, J, \omega)$ defined on $(M, J)$, the de Rham class $[\rho]$ of the corresponding Ricci form is equal to $2\pi c_1(M, J)$, where $c_1(M, J)$ stands for the (real) first Chern class of $(TM, J)$. The Calabi-Yau theorem can then be stated as follows

**Theorem 1.20.1 (Calabi-Yau theorem, 1st statement).** Let $(M, J)$ be a compact complex manifold admitting Kähler metrics and let $\Omega$ be any Kähler class. Then any (closed) $J$-invariant 2-form in the de Rham class $2\pi c_1(M, J)$ is the Ricci form of a unique Kähler metric in $\mathcal{M}_\Omega$.

A (real) $2m$-form $v$ on $M$ is called positive if $v$ has no zero and $\int_M v > 0$, when $M$ is oriented by $J$. In view of (1.19.7), the Calabi-Yau can then be re-stated in the following equivalent form

**Theorem 1.20.2 (Calabi-Yau theorem, 2nd statement).** Let $(M, J)$ be a compact complex manifold admitting Kähler metrics and let $\Omega$ be any Kähler class. Then any positive $2m$-form $v$ such that $\int_M v = \frac{\Omega^n}{m!} [M]$ is the volume form of a unique Kähler metric in $\mathcal{M}_\Omega$.

The equivalence of the two statements is established in the following manner. Let $\omega_0$ be the Kähler form of a fixed element $g_0$ in $\mathcal{M}_\Omega$, $\rho_0$ and $\psi_0$ the corresponding Ricci form and volume-form. By Proposition 1.17.2, a $J$-invariant 2-form $\psi$ belongs to the de Rham class $2\pi c_1(M, J)$ if and only if it is of the form $\psi = \rho_0 - \frac{1}{2} dd^c f$, for some real function uniquely defined...
Theorem 1.20.1 was first postulated by E. Calabi in [43], [44] and has since then be referred to as the \textit{Calabi conjecture}, until a complete proof was provided by S.-T. Yau in 1976 [197], [198]. The method of resolution

3 This inequality can be read $\frac{1}{m} \sum_{j=1}^{m} \log a_j \leq \log \frac{\sum_{j=1}^{m} a_j}{m}$, which is itself a specialization of the inequality $\sum_{j=1}^{m} t_j \log a_j \leq \log \sum_{j=1}^{m} t_j a_j$, for any $m$-uple $(t_j \geq 0$ with $\sum_{j=1}^{m} t_j = 1$; this, in turn, is an easy consequence of the concavity of the graph of the log function.
suggested by E. Calabi in [43] consists in introducing an auxiliary parameter \( t \) in \([0, 1]\) and in replacing the Monge-Ampère equation (1.20.2) by the following family, \( E(t) \), of Monge-Ampère equations:

\[
(\omega_0 + dd^c \phi)^m = e^{ft} \omega_0^m,
\]

where \( f_t \) is defined by \( e^{ft} = \frac{e^{ft}}{\int M e^{ft} v_0} \) (\( f_t \) is then a smooth curve joining \( f_0 = 1 \) to \( f \) in the space, say \( \mathcal{H}_0 \), of real functions on \( M \) normalized by (1.20.1)). Then, \( E(1) \) is the initial equation, whereas \( E(0) \) admits constant functions as obvious solutions.

Denote by \( S \subset [0, 1] \) the set of \( t \) such that \( E(t) \) admits a solution (this is then unique up to an additive constant, cf. Remark 1.20.1); \( S \) is not empty, as it contains 0, and Theorem 1.20.3 amounts to proving that \( S \) contains 1, in fact coincides with the whole segment \([0, 1]\), and this is done by showing that \( S \) is both open and closed in \([0, 1]\).

The proof that \( S \) is open in \([0, 1]\) can be outlined as follows. Denote by \( \mathcal{V} \) the map from \( M_\Omega \) to \( \mathcal{H}_0 \) defined by \( \mathcal{V} : g \mapsto \log \frac{g}{v_0} \); then, for any \((g, \omega)\) in \( M_\Omega \), the tangent space \( T_g M_\Omega \) is naturally identified to \( C^\infty(M, \mathbb{R})/\mathbb{R} \), hence to the space, \( C^\infty_{0,g}(M, \mathbb{R}) \), of functions in \( C^\infty(M, \mathbb{R}) \) which integrate to zero along \( v_g \) — cf. Section 3.2 — whereas the tangent space of \( \mathcal{H}_0 \) at \( \mathcal{V}(g) \) is also naturally identified to \( C^\infty_{0,g}(M, \mathbb{R}) \) (by differentiating (1.20.1) we get \( \int_M \hat{f} v_g = 0 \), which is the same as \( \int_M f v_g = 0 \) when \( f = \mathcal{V}(g) \)). The derivative of \( \mathcal{V} \) at \( \omega \) is then simply the map

\[
\hat{f} \mapsto -\Delta_g \hat{f},
\]

where \( \Delta_g \) is the riemannian Laplace operator relative to \( g \), cf. (3.2.4) in Chapter 3. Now, we already observed that \( \Delta_g \) is an isomorphism from \( C^\infty_{0,g}(M, \mathbb{R}) \) to itself, whose inverse is the Green operator \( G \) relative to \( g \), introduced in Section 1.17. By introducing appropriate Sobolev spaces and by using the elliptic regularity of \( \Delta_g \) — cf. Section 5.3 for details in a similar context — we infer that \( \mathcal{V} \) is a local diffeomorphism from \( M_\Omega \) to its image in \( \mathcal{H}_0 \). Now, if \( \phi_{t_0} \) is a solution to \( E(t_0) \) for some \( t_0 \) in \([0, 1]\) — meaning that \( \mathcal{V}(\omega_{t_0} + dd^c \phi_{t_0}) = f_{t_0} \) — then, for any \( t \) close enough to \( t_0 \), \( f_t \) is sufficiently close to \( f_{t_0} \) in the Fréchet topology of \( C^\infty_{0,g_0}(M, \mathbb{R}) \) so that \( f_t \) still lies in the image of \( \mathcal{V} \), i.e. \( E(t) \) admits solutions. This completes the (sketch of) proof that \( S \) is open in \([0, 1]\).

Proving that \( S \) is also closed, hence coincides with the whole segment \([0, 1]\), requires a priori estimates — independent of the parameter \( t \) — for solutions of \( E_t \) and their derivatives and has revealed a difficult question, solved by S.-T. Yau in [197]-[198]. Many variants and improvements of Yau’s original proof appeared since then in the literature, see e.g. [19], [173], [178], [185], [112], [33].

### 1.21. Kähler-Einstein metrics

A closely related problem concerns the search of Kähler-Einstein metrics, i.e. of Kähler structures \((g, J, \omega)\) such that \( g \) is an Einstein metric, on a fixed (connected) compact complex manifold \((M, J)\) (we implicitly assume that
\((M, J)\) is \textit{kählerian}, i.e. admits Kähler metrics). In terms of the Ricci form \(\rho\) and the Kähler class \(\omega\), the Kähler-Einstein condition is then

\begin{equation}
\rho = k\omega,
\end{equation}

where \(k\) is some real number, equal to \(\frac{\omega_0}{2\pi}\) where \(s\) is the (constant) scalar curvature. By considering the corresponding de Rham classes, this implies

\begin{equation}
c_1(M, J) = \frac{k}{2\pi}\Omega,
\end{equation}

which we already met in Proposition 1.19.2. This provides an obvious necessary condition for the existence of a Kähler-Einstein metric on \((M, J)\), namely that the Chern class \(c_1(M, J)\) be equal to zero, or positive — meaning that it is a Kähler class for some Kähler metric on \((M, J)\), or negative — meaning that \(-c_1(M, J)\) is positive.

Moreover, these conditions are pairwise incompatible: If \(c_1(M, J) = 0\), \(c_1(M, J)\) cannot contain a Kähler form \(\omega\), as \([\omega]^m[M] > 0\) for any Kähler class on \((M, J)\), nor the opposite of a Kähler form for the same reason. Suppose now that \(c_1(M, J)\) contains a Kähler form, say \(\omega\), and the opposite of a Kähler form, say \(-\omega\); then, \([\omega] = [\omega]\) implies \([\omega]^m[M] = -([\omega] \cup [\omega]^{-1})[M]\); but this cannot be, as \([\omega]^m[M]\) and \([\omega] \cup [\omega]^{-1})[M]\) are both positive (for the latter, observe that \(\omega\) can be diagonalized with respect to \(\omega\), with positive eigenvalues).

It follows that a (kählerian) compact complex manifold \((M, J)\) with \(c_1(M, J) = 0\) equal to zero, resp. positive, resp. negative, can only admit Kähler-Einstein metrics with \(k = 0\) — hence Ricci-flat — resp. \(k > 0\), resp. \(k < 0\), and admits no Kähler-Einstein if \(c_1(M, J)\) does not satisfy any of the above three conditions.

\textbf{Remark 1.21.1.} A compact complex manifold \((M, J)\) such that \(c_1(M, J)\) is positive or negative is a projective manifold, cf. Section 8.1. If \(c_1(M, J)\) is positive, \((M, J)\) is called a \textit{Fano manifold}.

If \(c_1(M, J) = 0\), then Theorem 1.20.1 guarantees the existence of a unique Ricci-flat Kähler metric in any Kähler class \(\Omega\) of \((M, J)\).

In view of this, we now assume that \(1.21.2\) is satisfied for some non-zero real number \(k\). The Kähler class \(\Omega\) is then well defined by \(1.21.2\) and, for any arbitrary chosen metric \(g_0\), of Kähler form \(\omega_0\), in \(\Omega\), the Ricci form \(\rho_{g_0}\) of \(\omega_0\) is of the form \(\rho_{g_0} = k\omega_0 - \frac{1}{2}dd^c p_{g_0}\), where \(p_{g_0}\) is the Ricci potential defined by \((1.19.10)-(1.19.11)\) in Section 1.19. By Proposition 1.17.2 the Kähler form of any other element in \(\Omega\) is of the form \(\omega_\phi = \omega_0 + dd^c \phi\), for some real function \(\phi\), well-defined up to an additive real constant. Denote by \(g_\phi\) the corresponding riemannian metric and by \(\rho_{g_\phi}\) the corresponding Ricci form. By \((1.19.7)\), \(\rho_{g_\phi}\) is given by \(\rho_{g_\phi} = \rho_{g_0} - \frac{1}{2}dd^c \log \frac{\omega_0 + dd^c \phi}{\omega_0} = k\omega_0 - \frac{1}{2}dd^c(p_{g_0} - \log \frac{\omega_0 + dd^c \phi}{\omega_0^m})\), whereas, because of \((1.21.2)\), the Ricci potential, \(p_{g_\phi}\), of \(g_\phi\) is determined by \(\rho_{g_\phi} = k\omega_\phi - \frac{1}{2}dd^c p_{g_\phi}\) and \(\int_M p_{g_\phi} v_{g_\phi} = 0\). We thus get:

\begin{equation}
p_{g_\phi} = p_{g_0} + 2k\phi + \log \frac{(\omega_0 + dd^c \phi)^m}{\omega_0^m},
\end{equation}
up to an additive real constant. It follows that \( \omega_\phi \) is the Kähler form of a Kähler-Einstein metric if and only if \( \phi \) is a solution to the following Monge-Ampère equation:

\[
(\omega_0 + \ddc \phi)^m = e^{-\tilde{p}_{g_0} - 2k\phi - c(\phi)} \omega_0^m,
\]

for some appropriate constant \( c(\phi) \) depending upon a chosen normalization of \( \phi \) and also on the chosen normalization (1.19.11) of the Ricci potential \( p_{g_0} \). The most common option consists in substituting \( \tilde{p}_{g_0} = p_{g_0} + \log \frac{1}{V_{\Omega}} \int_M e^{-p_{g_0}} v_{g_0} \) to \( p_{g_0} \)— where \( V_{\Omega} = \frac{\Omega^m}{m!} [M] \) denotes the common total volume \( \int_M v_g \) for all metrics \( g \in \mathcal{M}_\Omega \)— so as to have the alternative normalization for the Ricci potential of \( g_0 \):

\[
\frac{1}{V_{\Omega}} \int_M e^{-\tilde{p}_{g_0}} v_{g_0} = 1,
\]

and in substituting the following Monge-Ampère equation

\[
(\omega_0 + \ddc \phi)^m = e^{-\tilde{p}_{g_0} - 2k\phi} \omega_0^m,
\]

where \( \phi \) is normalized \textit{a posteriori} by the obvious necessary condition:

\[
\frac{1}{V_{\Omega}} \int_M e^{-\tilde{p}_{g_0} - 2k\phi} v_{g} = 1
\]

(for an alternative option, see Chapter 7).

Remark 1.21.2. By allowing \( k = 0 \) in (1.21.6) and by replacing \( -\tilde{p}_{g_0} \) by any real function \( f \) normalized by (1.20.1), we recover the Monge-Ampère equation (1.20.2) associated to the general Calabi conjecture.

Like for the Monge-Ampère equation (1.20.2), the strategy of resolution to (1.21.6), whith \( k \neq 0 \), relies on the \textit{continuity method}, which, for \( k \neq 0 \), consists in substituting the following family of Monge-Ampère equations, parametrized by \( t \) in \([0,1]\):

\[
(\omega_0 + \ddc \phi)^m = e^{-\tilde{p}_{g_0} - 2k t \phi} \omega_0^m,
\]

where, again, \( \phi \) is normalized \textit{a posteriori} by

\[
\frac{1}{V_{\Omega}} \int_M e^{-\tilde{p}_{g_0} - 2k t \phi} v_{g_0} = 1.
\]

As in Section 1.20, denote by \( S \) the set of \( t \) in \([0,1]\) such that (1.21.8) has a solution. By Theorem 1.20.3, 0 belongs to \( S \), which is then non-empty. As in the case when \( k = 0 \), it is easy to check that \( S \) is open (cf. Proposition 7.3.2 in Chapter 7 when \( k > 0 \); the case when \( k < 0 \) is similar, even easier. see [17]), whereas the hard work starts when trying to prove that \( S \) is closed, hence coincides with the whole closed interval \([0,1]\). This was done by T. Aubin, cf. also [197] [198], in the case when \( k \) is negative. We then have

\textbf{Theorem 1.21.1 (T. Aubin [17], [18])}. let \((M,J)\) be a compact complex manifold, whose first \textit{real} Chern class \( c_1(M,J) \) is negative. Then, \((M,J)\) admits a Kähler-Einstein metric — necessarily of negative scalar curvature — unique up to scaling.
Proof. Again, the proof of the existence part of Theorem 1.21 goes beyond the scope of this book. The uniqueness part amounts to showing that the following Monge-Ampère equation

\[(ω + dd^c φ)^m = e^{-2k φ} ω^m,\]

has φ ≡ 0 as unique solution for any k < 0. This is done as follows. First notice that (1.21.10) forces φ to satisfy the condition

\[(1.21.11) \int_M e^{-2k φ} ω^m = \int_M ω^m.\]

Since k < 0, it follows from (1.21.11) that φ(x_M) ≥ 0 at each point x_M where φ attains its maximum and that φ ≡ 0 whenever φ(x_M) = 0. Now, by an argument already used in Remark 1.20.1, we have that

\[(1.21.12) \Delta φ ≤ m(1 - e^{-2k φ(x_M)}),\]

where Δ denotes the riemannian laplacian relative to the chosen reference metric associated to ω. We thus get 0 ≤ Δφ(x_M) ≤ m(1 - e^{-2k φ(x_M)}) ≤ 0, hence φ(x_M) = 0, hence φ ≡ 0. □

When k > 0, neither the uniqueness, nor the existence of a solution to the Monge-Ampère equation (1.21.4) can be expected in general. The uniqueness of a solution may fail because of the possible existence of automorphisms of (M, J) in the identity component which don’t preserve the Kähler-Einstein metric. In view of Remark 1.23.2, this cannot happen if k = 0 or k < 0, but may certainly happen if k > 0, e.g. if (M, J) is a complex projective space. On the other hand, it is a deep theorem of S. Bando and T. Mabuchi [23] that Kähler-Einstein metrics on a Fano manifold (M, J), if any, all belong to a single orbit of \(H_{\text{red}}\) in each \(M_{Ω, Ω = 2π k c_1(M, J), k > 0}\). According to an old result of Y. Matsushima, the existence fails whenever \(H(M, J)\) is not the complexification of one of its maximal compact subgroups, cf. Theorem 3.6.2 in Section 3.6 below. More information on these questions in the Fano case can be found in Section 6.6 and in Chapter 7 below.

1.22. Bochner identities and vanishing theorems

For any riemannian manifold (M, g), the Bochner formula for 1-forms is written as

\[(1.22.1) \Delta α = δDα + r(α^\dagger),\]

for any 1-form α, where \(Δ = δd + dδ\) denotes the riemannian Laplace operator (here acting on 1-forms), \(D = D^g\) the Levi-Civita connection of g and r stands for the Ricci tensor of g — cf. Section 1.18 — viewed as an homomorphism from \(TM\) to \(T^*M\). There is a similar Bochner formula for (scalar) p-forms and, more generally, for \(E\)-valued p-forms, for any vector bundle E equipped with a fiberwise positive definite inner product and a compatible linear connection \(∇\) (if E is a complex fiber bundle, we implicitly assume that the inner product is hermitian and that \(∇\) is a \(\mathbb{C}\)-linear connection). In the latter case, \(D^∇\) will denote the linear connection induced by \(∇\) and the Levi-Civita connection on the vector bundle \(E \otimes Λ^*M\).
of $E$-valued exterior forms. As in Sections 1.6 and 1.10, we denote by $d^\nabla$, $\delta^\nabla$ and $\Delta^\nabla = \delta^\nabla d^\nabla + d^\nabla \delta^\nabla$ the exterior derivative, the codifferential and the Laplace operator relative to $\nabla$ and the metric $g$. The general Bochner formula also involves the so-called rough Laplace operator $(D^\nabla)^* D^\nabla$, which can also be written as $\delta^\nabla D^\nabla$ — cf. Remark 1.10.1. We then have

**Proposition 1.22.1** (General Bochner formula). With the above hypotheses, the Laplace operators $\Delta^\nabla$ and $\delta^\nabla D^\nabla$ are related by

\begin{equation}
\Delta^\nabla \Phi = \delta^\nabla D^\nabla \Phi + R_p^\nabla(\Phi),
\end{equation}

for any $E$-valued $p$-form $\Phi$, where $R_p^\nabla(\Phi)$ — a universal linear expression in $\Phi$ and the curvature, $R^D$, of $D^\nabla$ — has the following expression

\begin{equation}
(R_p^\nabla(\Phi))_{X_1,\ldots,X_p} = \sum_{k=1}^p \sum_{i=1}^n (R^D_{X_k,e_i}(\Phi))_{X_1,\ldots,e_i,\ldots,X_p} = \sum_{k=1}^p R^\nabla_{X_k,e_i}(\Phi)_{X_1,\ldots,e_i,\ldots,X_p} + \sum_{k=1}^p \Phi_{X_1,\ldots,e_iX_k,\ldots,X_p} - \sum_{1 \leq r < s \leq p} (-1)^{r+s} \sum_{i=1}^n \Phi_{X_r,e_i,e_iX_s,\ldots,X_p}
\end{equation}

for any vector fields $X_1,\ldots,X_p$ and any auxiliary orthonormal local frame $\{e_1,\ldots,e_n\}$ of $TM$; in the rhs of (1.22.2) it is understood that the second $e_i$ sits in place of $X_k$.

**Proof.** It is convenient here to introduce the second covariant derivative $D^\nabla^2 \Phi := D^\nabla(D^\nabla \Phi)$, acting on $E$-valued exterior forms $\Phi$, which is the covariant derivative of $D^\nabla \Phi$, viewed as a section of $T^*M \otimes (E \otimes \Lambda^*M)$ relative to the connection induced by $\nabla$ and $D = D^\nabla$. For any decomposed $E$-valued exterior forms $\Phi = s \otimes \psi$ and for any vector fields $X,Y$ on $M$, we have:

\begin{equation}
(D^\nabla^2(s \otimes \psi))_{X,Y} = \nabla^2_{X,Y}s \otimes \psi + +s \otimes D^2_{X,Y}\psi + \nabla Xs \otimes D_Y \psi + \nabla Ys \otimes D_X \psi.
\end{equation}

It follows that:

\begin{equation}
R^D_{X,Y} \Phi = -D^\nabla_{X,Y} \Phi + D^\nabla_{Y,X} \Phi,
\end{equation}

whereas

\begin{equation}
\delta^\nabla D^\nabla \Phi = -\sum_{i=1}^n D^\nabla_{e_i,e_i} \Phi.
\end{equation}

From (1.6.4)-(1.10.14), we then get

\begin{equation}
(\delta^\nabla d^\nabla \Phi)_{X_1,\ldots,X_p} = -\sum_{i=1}^n (D^\nabla_{e_i} d^\nabla \Phi)_{e_i,X_1,\ldots,X_p} = -\sum_{i=1}^n (D^\nabla_{e_i,e_i} \Phi)_{X_1,\ldots,X_p} - \sum_{k=1}^p (-1)^{k-1} \sum_{i=1}^n (D^\nabla_{e_iX_k} \Phi)_{e_i,X_1,\ldots,X_k,\ldots,X_p},
\end{equation}
and

\[(d^\nabla \delta^\nabla \Phi)_{X_1,\ldots,X_p} = \sum_{k=1}^{p} (-1)^{k-1} (D_{X_k} \delta^\nabla \Phi)_{X_1,\ldots,X_k,\ldots,X_p}\]

(1.22.8)

By adding (1.22.7) and (1.22.8), and by using (1.22.5) and (1.22.6), we readily get the first expression in the rhs of (1.22.2). The second expression in the rhs of (1.22.2) easily follows by using the algebraic Bianchi identity (1.18.2).

In the Kähler setting, we infer the following general Bochner-like formula: Akizuki-Nakano identity (1.16.3), we get

\[2 \Box^E \Phi = \delta^\nabla D^\nabla \Phi + i [\Lambda, R^\nabla] \Phi + R^\nabla p(\Phi),\]

for any \(E\)-valued \(p\)-form \(\Phi\).

**Proof.** Direct consequence of Proposition 1.22.1 and of the Akizuki-Nakano identity (1.16.3).

**Remark 1.22.1.** In the case when \(\Phi\) is a scalar \(p\)-form — then denoted \(\psi\) — \(\Delta^\nabla\) is the usual riemannian Laplace operator, \(D^\nabla = D\) is the Levi-Civita connection and \(R^D^\nabla\) is then simply the riemannian curvature acting on \(p\)-forms, \(R^\nabla p(\Phi)\) will be simply denoted \(R_p(\psi)\) and the above expression reduces to

\[R_p(\psi)_{X_1,\ldots,X_p} = + \sum_{k=1}^{p} \Phi_{X_1,\ldots,r(X_k),\ldots,X_p} - \sum_{1 \leq r < s \leq p} (-1)^{r+s} \sum_{i=1}^{n} \Phi_{R,\ldots,\hat{X}_r,\ldots,\hat{X}_s,\ldots,X_p}.\]

In particular, (1.22.2) reduces to (1.22.1) for scalar 1-forms. Notice that (1.22.10) can in turn be rewritten in the following closer form:

\[R_p(\psi) = - \sum_{1 \leq i < j \leq n} (e_i \wedge e_j) \cdot R_{e_i,e_j}\psi.\]

This expression involves the induced (skew symmetric) action — here denoted by a dot \(\cdot\) — of the bundle, \(A(M)\), of skew-symmetric endomorphisms of \(TM\) on \(\Lambda^pM\), defined by

\[(A \cdot \psi)_{X_1,\ldots,X_p} = - \sum_{r=1}^{p} \psi_{X_1,\ldots,A_r,\ldots,X_p},\]

for any section \(A\) of \(A(M)\): at each point \(x\) of \(M\) this action is an action of the Lie algebra \(A_x(M)\) of anti-symmetric endomorphisms of \(T_xM\), which is
the Lie algebra of the group $O(T_xM)$ of orthogonal automorphisms of $T_xM$, and (1.22.12) is then the natural induced action on $\Lambda^pT_xM = \Lambda^p(T_x^\ast M)$. In (1.22.11), $A(M)$ is identified with $\Lambda^2(TM)$ via the riemannian duality and this action then reads: $(X \wedge Y) \cdot \psi = -X^i \wedge (Y^j \cdot \psi) + Y^i \wedge (X^j \cdot \psi)$ for any vector fields $X, Y$ (it is then easy to deduce (1.22.11) from (1.22.10)); moreover, for any vector fields $X, Y$, $R_{X,Y}$ is a section of $A(M)$, also written $R(X \wedge Y)$ — cf. Section 1.18 — and we then have: $R_{X,Y} \cdot \psi = R(X \wedge Y) \cdot \psi$; the rhs of (1.22.11) can then be read as $-\sum_{1 \leq i < j \leq n} (e_i \wedge e_j) \cdot (R(e_i \wedge e_j) \cdot \psi)$; in this expression, the $e_i \wedge e_j$’s may be replaced by any orthonormal frame of $\Lambda^2(TM)$, in particular by an orthonormal frame of eigenforms $\omega_\kappa$ of $R$, for $\kappa = 1, \ldots, N = \frac{n(n-1)}{2}$; if $\lambda^R_1, \ldots, \lambda^R_N$ denote the corresponding (real) eigenvalues, we thus get

\begin{equation}
\mathcal{R}(\psi) = -\sum_{\kappa=1}^{N} \lambda^R_\kappa \cdot (\omega_\kappa \cdot \psi). \tag{1.22.13}
\end{equation}

The main interest of the Bochner formula (1.22.2) is to provide vanishing theorems. In the remainder of this section, we assume that $M$ is compact.

**Proposition 1.22.3.** Let $(M, g)$ be a connected, compact riemannian manifold of dimension $n$. For any integer $p$, $0 \leq p \leq n$, denote by $b_p$ the $p$-th Betti number of $M$.

(i) If the Ricci tensor $\kappa$ is everywhere semi-positive, as a symmetric endomorphism of $TM$, then all $g$-harmonic 1-forms are $D^g$-parallel; in particular, $b_1 \leq n$. If, moreover, $\kappa$ is positive at some point, then $b_1 = 0$.

(ii) If the riemannian curvature $R$ is everywhere semi-positive, as a symmetric endomorphism of $\Lambda^2M$, then all $g$-harmonic $p$-forms are $D^g$-parallel for any $p$; in particular, $b_p \leq \binom{n}{p}$ for any $p$. If, moreover, $R$ is positive at some point, then $b_p = 0$ for $0 < p < n$.

**Proof.** (i) For scalar 1-forms, the Bochner formula reduces to (1.22.1); when $M$ is compact, this implies $\langle \Delta \alpha, \alpha \rangle = \langle \delta D \alpha, \alpha \rangle + \langle \kappa(\alpha), \alpha \rangle$; since $\delta$ is a formal adjoint of $d$ and of $D$ — cf. Remark 1.10.1 — we infer $\langle d \alpha, d \alpha \rangle + \langle \delta \alpha, \delta \alpha \rangle = (D \alpha, D \alpha) + \langle \kappa(\alpha), \alpha \rangle$, for any (real) 1-form $\alpha$. If $\alpha$ is $g$-harmonic, we then get $0 = \langle D^g \alpha, D^g \alpha \rangle + \langle \kappa(\alpha), \alpha \rangle$. If $\kappa$ is everywhere non-negative as an endomorphism of $TM$ — hence of $T^*M$ by riemannian duality — both terms are non-negative, hence equal to zero; in particular, $D^g \alpha$ vanishes identically. Since $M$ is connected, $\alpha$ is then determined by its value at any point of $M$; it follows that $b_1 \leq n$. If $\kappa$ is positive at some point $x$, then $\alpha(x) = 0$ and $\alpha$ is then identically zero; we then have $b_1 = 0$.

(ii) For scalar $p$-forms the general Bochner formula (1.22.2) reduces to

\begin{equation}
\Delta \psi = \delta D \psi + \mathcal{R}_p(\psi), \tag{1.22.14}
\end{equation}

where $\mathcal{R}_p$ is given by (1.22.10), (1.22.11) or (1.22.13). From (1.22.13), we readily infer $\langle \mathcal{R}_p(\psi), \psi \rangle = \sum_{\kappa=1}^{N} \lambda^R_\kappa \cdot |\omega_\kappa \cdot \psi|^2$ (recall that the action (1.22.12) is anti-symmetric). If $R$ is non-negative — meaning that all the eigenvalues $\lambda^R_\kappa$ are non-negative — $\mathcal{R}_p$ is then non-negative for any $p$ (notice that $\mathcal{R}_0$ and $\mathcal{R}_n$ are identically zero). Moreover, if $R$ is positive at some point $x$, then $\mathcal{R}_p$ is also positive at $x$ for any $p$, $0 < p < n$. Indeed, $R$ is positive at $x$ if and only if the $\lambda^R_\kappa(x)$ are all positive; it follows that $\mathcal{R}_p(\psi)(x) \neq 0$, unless
\[ \omega_k \cdot \psi(x) = 0 \text{ for all } k = 1, \ldots, N. \] This in turns means that \( A \cdot \psi(x) = 0 \), for any \( A \) in the Lie algebra \( \mathfrak{a}_x(M) \) of anti-symmetric endomorphisms of \( T_xM \). Equivalently, \( \psi(x) \) is a fixed element of the action of \( SO(T_xM) \), cf. above. It is now an elementary and well-known fact that the (special) orthogonal group \( SO(n) \) acting on \( \Lambda^p \mathbb{R}^n \) has no non-zero fixed element for \( 0 < p < n \). Because of (1.22.14), the rest of the argument is as in (i): if the curvature operator \( R \) is everywhere non-negative, then \( R_p \) is everywhere non-negative for any \( p \); any harmonic \( p \)-form \( \psi \) is then \( Dg \)-parallel. If, moreover, \( R \) is positive at some point \( x \), then \( R_p \) is positive at \( x \), unless \( p = 0 \) or \( p = n \).

Then, \( \psi(x) = 0 \) and \( \psi \) is then identically zero; we then get \( b_p = 0 \), for any \( p \neq 0, n \). □

Remark 1.22.2. The Bochner formula (1.22.1) for scalar 1-forms can be refined as follows. For any (real) 1-form \( \alpha \), decompose the covariant \( Dg \alpha \) — which is a bilinear form — into its anti-symmetric and symmetric parts: we then get \( D\alpha = \frac{1}{2}d\alpha - \frac{1}{n}\delta\alpha g + \frac{1}{2}(\mathcal{L}_\alpha g)0 \), (easy verification). The symmetric part \( \frac{1}{2}\mathcal{L}_\alpha g \) can furthermore be decomposed into its trace part, equal to \( -\frac{1}{n}\delta\alpha g \) and its trace-free part, which we denote by \( \frac{1}{2}(\mathcal{L}_\alpha g)0 \). We eventually get the following decomposition:

\[
\text{(1.22.15)} \quad D\alpha = \frac{1}{2}d\alpha - \frac{1}{n}\delta\alpha g + \frac{1}{2}(\mathcal{L}_\alpha g)0,
\]
where the three components are pairwise orthogonal. The vector fields \( X \) such that \( (\mathcal{L}_X g)0 = 0 \) are called \textit{conformal Killing vector fields}: the space of conformal Killing vector fields is formally the Lie algebra of the group of diffeomorphisms of \( M \) which preserve the conformal structure \([g] \) determined by \( g \); since \( \mathcal{L}_X g = (\mathcal{L}_X g)_0 - \delta X^\flat \frac{n}{n} g \), a vector field \( X \) is Killing if and only if it is a divergence-free, conformal Killing vector field. By using (1.22.15), the Bochner formula (1.22.1) can be rewritten as:

\[
\text{(1.22.16)} \quad \frac{1}{2}d\alpha + \frac{(n-1)}{n}d\delta\alpha - \frac{1}{2}\delta(\mathcal{L}_\alpha g)_0 = r(\alpha^*),
\]
for any (real) 1-form \( \alpha \). We obtain the following proposition, which is the “negative counterpart” of Proposition 1.22.3 (i):

**Proposition 1.22.4.** Let \((M, g)\) be a connected, compact riemannian manifold of dimension \( n \). If the Ricci tensor \( r \) is everywhere semi-negative, then all conformal Killing vector fields are \( Dg \)-parallel; if, moreover, \( r \) is negative at some point, then the space of conformal Killing vector fields is reduced to zero.

**Proof.** Let \( X \) be a conformal Killing vector field and denote \( \xi = X^\flat \); by (1.22.16) we then have

\[
\frac{1}{2}\delta d\xi + \frac{(n-1)}{n}d\delta\xi = r(X);
\]
since \( M \) is compact, we infer \( \frac{1}{2}(d\xi, d\xi) + \frac{(n-1)}{n}(\delta\xi, \delta\xi) = \langle r(X), X \rangle \). If \( r \) is everywhere semi-negative, we infer that \( d\xi \) and \( \delta\xi \) both vanish identically, so that \( Dg\xi \equiv 0 \); if moreover \( r \) is negative at some point, \( \xi \) must vanish at this point, hence everywhere. □
1.22. Bochner Identities and Vanishing Theorems

In general, for any holomorphic vector bundle $E$ over a (compact) complex manifold $(M, J)$ and for any non-negative integers $p, q$, we denote by $H^{p,q}(M, E)$ the $q$-th cohomology group of the sheaf of holomorphic germs of sections of the holomorphic vector bundle $E \otimes \Lambda^{p,q}M$. By the classical Hodge-Kodaira-Dolbeault theory, $H^{p,q}(M, E)$ is isomorphic to the the space of $\Box^E$-harmonic $E$-valued forms of type $(p, q)$, for any Kähler metric $g$ on $M$ and any hermitian inner product $h$ on $E$, cf. e.g. [100, Chapter 1]. We then have the following classical vanishing theorems:

**Proposition 1.22.5** (Kodaira-Nakano-Akizuki vanishing theorem). Let $(M, J)$ be a connected, compact Kähler manifold of real dimension $n = 2m$. Let $L$ be any ample holomorphic line bundle over $M$. Then,

\[(1.22.17) \quad H^{p,q}(M, L) = \{0\}, \quad \forall p, q, \quad p + q > m.\]

In particular, if $K_M$ denotes the canonical line bundle of $M$, cf. Section 1.9, we have

\[(1.22.18) \quad H^q(M, K_M \otimes L) = \{0\}, \quad \forall q > 0.\]

(ii) For any holomorphic vector bundle $E$, there is a positive integer $k_0$ such that

\[(1.22.19) \quad H^q(M, E \otimes L^k) = \{0\}, \quad \forall q > 0, \quad \forall k \geq k_0.\]

**Proof.** (i) Since $L$ is ample, we can choose $h$ so that $R^i = i\omega$, where $\omega$ denotes the Kähler form of $M$. From (1.16.3), and by using (1.13.4)–(1.13.5), we then infer

\[(1.22.20) \quad \Box^L \Phi = \Box^\nabla \Phi - [\Lambda, L] \Phi = \Box^\nabla \Phi + (p + q - m) \Phi,\]

for any $L$-valued form $\Phi$ of type $(p, q)$. By using the hermitian inner product on $L$-forms induced by $g$ and $h$, we infer

\[(1.22.21) \quad \int_M (\Box^L \Phi, \Phi) v_g = \int_M (\Box^\nabla \Phi, \Phi) v_g + (p + q - m) \int_M |\Phi|^2 v_g,\]

where $\int_M (\Box^L \Phi, \Phi) v_g$ and $\int_M (\Box^\nabla \Phi, \Phi) v_g$ are both non-negative. If $\Phi$ is $\Box^L$-harmonic and $(p + q > m$, we thus get $\Phi = 0$, hence (1.22.17); (1.22.18) readily follows, since $H^q(M, K(M) \otimes L) = H^{m,q}(M, L)$.

(ii) It is convenient to write $E \otimes L^k$ as $K_M \otimes (\tilde{E} \otimes L^k)$, by setting $\tilde{E} = E \otimes K^{-1}(M)$, so that $H^q(M, E \otimes L^k) = H^{m,q}(M, \tilde{E} \otimes L^k)$. If $L$ is equipped with the Chern connection as above and $\tilde{E}$ with any Chern connection, the resulting Chern curvature, $R^{\tilde{E} \otimes L^k}$ say, on $\tilde{E} \otimes L^k$ is of the form $R^{\tilde{E} \otimes L^k} = i k \omega \otimes \text{Id}_{\tilde{E} \otimes L^k} + R^{\tilde{E}} \otimes \text{Id}_{L^k}$, where $\omega$ denotes the Kähler form of the chosen Kähler metric on $M$, so that the hermitian operator $i [\Lambda, R^{\tilde{E} \otimes L^k}]$, acting on the vector bundle $(\tilde{E} \otimes L^k) \otimes \Lambda^{0,q}M$, is of the form $-k [\Lambda, L] \otimes \text{Id}_{\tilde{E} \otimes L^k} + i [\Lambda, R^{\tilde{E}}] \otimes \text{Id}_{L^k} = k q \text{Id}_{\tilde{E} \otimes L^k} + i [\Lambda, R^{\tilde{E}}] \otimes \text{Id}_{L^k}$. If $q > 0$, there clearly exists a positive integer $k_0$ such that the hermitian operator $k q \text{Id} + i [\Lambda, R^{\tilde{E}}]$ be positive definite at each point of $M$ — as $M$ is compact — and, we then infer from (1.16.3) as before that $H^{m,q}(M, \tilde{E} \otimes L^k) = H^q(M, E \otimes L^k) = \{0\}$ for any integer $k \geq k_0$. \(\square\)
Proposition 1.22.6. Let \((M, g, J, \omega)\) a connected, compact almost hermitian manifold and let \((E, \bar{\partial} E, h)\) be a hermitian (almost) holomorphic vector bundle over \(M\). Denote by \(\nabla\) the corresponding Chern connection — cf. Proposition 1.6.1 — and by \(R^\nabla\) the curvature of \(\nabla\). Define \(K^\nabla := -2i\Lambda R^\nabla\): \(K^\nabla\) is then a hermitian endomorphism, called the Chern endomorphism of \(E\) with respect to \(\omega\). Call holomorphic any section of \(E\) which belongs to the kernel of \(\bar{\partial} E\). We then have:

\[
-\Lambda dd^c |s|^2 = (K^\nabla(s), s) - 2|\nabla s|^2,
\]

for any holomorphic section \(s\) of \(E\). In particular, if \(K^\nabla\) is everywhere semi-negative, then all holomorphic sections of \(E\) are \(\nabla\)-parallel. If, moreover, \(K^\nabla\) is negative at some point of \(M\), then \(E\) admits no non-zero holomorphic section.

**Proof.** Let \(s\) be any holomorphic section of \(E\); denote by \((\cdot, \cdot)\) the hermitian inner product \(h\) and by \(|s|^2 = (s, s)\) the square norm of \(s\) with respect to \(h\). We then have \(dd^c |s|^2(X) = -\partial_X s - \partial_Y s\), where \(\partial_X s = i(\partial_X s, \partial_Y s)\) for any (real) vector field \(X\) (notice that \(\nabla_X s = i\nabla_X s\), as \(s\) is holomorphic and \(\bar{\partial} E = \nabla h\)). We readily infer \(dd^c |s|^2(X, Y) = 2i(\nabla_X |s|^2, s) + 2i(\nabla_Y |s|^2, s)\), for any (real) fields \(X, Y\). Now \(-\Lambda dd^c |s|^2 = -\frac{1}{2} \sum_{i=1}^n dd^c |s|^2(e_i, Je_i)\), for any auxiliary orthonormal frame \(\{e_1, \ldots, e_n\}\), and we thus get (1.22.22). If \(K^\nabla\) is semi-negative, the rhs of (1.22.22) is non-positive and we then have \(-\Lambda dd^c |s|^2 \leq 0\) everywhere. Since \(M\) is compact, this implies that \(|s|^2\) is constant — cf. [103] — so that both sides of (1.22.22) vanish identically. This implies \(\nabla s \equiv 0\) and \((K^\nabla s, s) \equiv 0\); moreover, if \(K^\nabla\) is negative at some point of \(M\), then \(s\) vanishes at this point, hence is identically zero, as it is \(\nabla\)-parallel.

1.23. The Lichnerowicz fourth order operator

On any almost complex manifold \((M, J)\), a (real) vector field is said to be (real) holomorphic if it is a holomorphic section of \((TM, J)\), i.e. if

\[
\mathcal{L}_J = 0,
\]

where, we recall, \(\mathcal{L}_X\) denotes the Lie derivative along \(X\). Since \([\mathcal{L}_X, \mathcal{L}_Y] = [\mathcal{L}_X, \mathcal{L}_Y]\), the space of (real) holomorphic vector fields is closed for the bracket, hence is a Lie subalgebra of the space of all (real) vector fields. It will be denoted by \(\mathfrak{h}(M, J)\) or simply \(\mathfrak{h}\) when \((M, J)\) is understood.

This is only a real Lie algebra in general, but when \(J\) is integrable the action of \(J\) turns it into a complex Lie algebra. Indeed, the integrability of \(J\) ensures that \(JX\) belongs to \(\mathfrak{h}\) as soon as \(X\) does and we then have \([JX, JY] = J[JX, Y] = -(\mathcal{L}_Y J)(X) = 0\) for all pairs of elements \(X, Y\) of \(\mathfrak{h}\), cf. Lemma 1.1.1.

**Remark 1.23.1.** As already observed in Section 1.8, a (real) vector field \(X\) on a complex manifold \((M, J)\) is (real) holomorphic if and only if its \((1, 0)\)-part \(X^{1,0} = \frac{1}{2}(X - iJX)\) is holomorphic in the usual sense as a section of the (holomorphic) vector bundle \(T^{1,0}M\).
If, moreover, $g$ is a Kähler riemannian metric on $(M, J)$, of Levi-Civita connection $D$, (real) holomorphic vector fields can be alternatively characterized as follows

**Lemma 1.23.1.** A (real) vector field $X$ is holomorphic if and only if the following condition is satisfied:

$$D_JYX = JD_YX,$$

for any vector field $Y$.

**Proof.** Lemma 1.23.1 is a direct consequence of Proposition 1.8.1. An alternative derivation is as follows. For any field of endomorphisms $A$ and any torsion-free connection $D$ the following general identity holds:

$$\mathcal{L}_XA = DXA - [DX,A],$$

for any vector field $X$ (here, $DX$ is viewed as the endomorphism: $Y \rightarrow D_YX$. The fact that $D$ is torsion-free can then be written as $\mathcal{L}_XZ = DXZ - DX(Z)$, for any pair $X, Z$ of vector fields; this readily implies (1.23.3).

For a general almost hermitian structure $(g, J, \omega)$ and for any vector field $X$, we then have

$$\mathcal{L}_XJ = DXJ - [DX,J].$$

In the Kähler case, $DJ = 0$ so that $X$ is a (real) holomorphic vector field if and only if

$$[DX,J] = 0;$$

this is the same as (1.23.2).

For any real 1-form $\alpha$ we denote by $D^+\alpha$, resp. $D^-\alpha$, the $J$-invariant part, resp. the $J$-anti-invariant part, of the covariant derivative $D\alpha$ (as a bilinear form); we thus have:

$$(D^\pm\alpha)_{X,Y} = \frac{1}{2}((DX\alpha)Y \pm (DJX\alpha)JY).$$

In the remainder of this section, the structure is assumed to be Kähler. In terms of Lie derivative, $D^-$ can then be expressed as follows:

**Lemma 1.23.2.** For any real 1-form $\alpha$, we have:

$$D^-\alpha = -\frac{1}{2}g(J(\mathcal{L}_\alpha J)\cdot,\cdot) = -\frac{1}{2}\omega((\mathcal{L}_\alpha J)\cdot,\cdot).$$

In particular, $\alpha$ is the dual 1-form of a holomorphic (real) vector field if and only if $D^-\alpha = 0$.

**Proof.** Simple reformulation of (1.8.3)-(1.8.4).

**Lemma 1.23.3.** For any real 1-form $\alpha$, we have

$$\delta D^+\alpha - \delta D^-\alpha = r(\alpha^\perp).$$
Proof. For any vector fields $X, Y$, we have that $\langle (D^+ - D^-)\alpha \rangle_{X,Y} = (D\alpha)_{JX,JY}$, and then:

$\langle \delta(D^+ - D^-)\alpha \rangle(X) = -\sum_{j=1}^{n}(D_{e_j}^2, e_j)(JX) = -\frac{1}{2}\sum_{j=1}^{n}(D_{e_j}^2, e_j - D_{\overline{e}_j}^2, e_j)(JX) = \frac{1}{2}\sum_{j=1}^{n}(R_{e_j} e_j)(JX) = \rho(\alpha^\sharp, JX) = r(\alpha^\sharp, X)$. □

**Lemma 1.23.4.** For any real 1-form $\alpha$, we have:

\begin{align*}
\delta D^+ \alpha &= \frac{1}{2}\Delta \alpha, \\
\delta D^- \alpha &= \frac{1}{2}\Delta \alpha - r(\alpha^\sharp) = \frac{1}{2}\Delta \alpha + J\alpha^\sharp \rho.
\end{align*}

If $M$ is compact, $\alpha$ is harmonic if and only if $D^+ \alpha = 0$, and $\alpha$ is the dual of a (real) holomorphic vector field, $X = \alpha^\sharp$, if and only if it satisfies

\begin{equation}
\frac{1}{2}\Delta \alpha - r(\alpha^\sharp) = \frac{1}{2}\Delta \alpha + J\alpha^\sharp \rho = 0.
\end{equation}

Proof. In terms of the operators $D^\pm$, the Bochner formula (1.22.1) reads:

\begin{equation}
\delta D^+ \alpha + \delta D^- \alpha = \Delta \alpha - r(\alpha^\sharp).
\end{equation}

Then, (1.23.9)–(1.23.10) readily follow from (1.23.12)–(1.23.8). If $M$ is compact, we have that $\langle \delta D^\pm \alpha, \alpha \rangle = \langle D^\pm \alpha, D\alpha \rangle = \langle D^\pm \alpha, D\alpha \rangle$, so that $\delta D^\pm \alpha = 0$ if and only if $D^\pm \alpha = 0$. The second assertion then readily follows from (1.23.9)–(1.23.10) and from Lemma 1.23.2. □

**Remark 1.23.2.** The characterisation of holomorphic vector fields on compact Kähler manifolds given in Lemma 1.23.4 is due to A. Lichnerowicz, cf. [135]. It readily implies that a compact Kähler manifold whose Ricci tensor $r$ is negative definite admits no non-trivial holomorphic vector field, and that if $r$ is semi-negative any holomorphic vector field is parallel with respect to the Levi-Civita connection.

**Lemma 1.23.5.** For any real 1-form $\alpha$, we have:

\begin{align*}
\delta \delta D^- \alpha &= \frac{1}{2}\Delta \delta \alpha - (d^c \alpha, \rho) + \frac{1}{2}(\alpha, ds), \\
L(f) := \delta \delta D^- df &= \frac{1}{2}\Delta^2 f + (dd^c f, \rho) + \frac{1}{2}(df, ds),
\end{align*}

In particular, for any function $f$, we have

\begin{align*}
\delta \delta D^- df &= \frac{1}{2}\Delta^2 f + (dd^c f, \rho) + \frac{1}{2}(df, ds).
\end{align*}
and

\[ \delta \delta D^{-} d^c f = -\frac{1}{2} \mathcal{L}_K f, \]

where \( \mathcal{L}_K \) denotes the Lie derivative along the vector field \( K := J \text{grad}_g s \).

**Proof.** \((1.23.13)\) follows from \((1.23.10)\) and the adjunction formula — cf. \((1.10.12)\) —

\[ \delta(\langle X \lrcorner \Psi \rangle) = (\langle dX, \Psi \rangle) - (X \lrcorner \delta \Psi), \]

for any vector field \( X \) and any 2-form \( \Psi \); \((1.23.14)\) and \((1.23.15)\) follow easily by specializing \( \xi = df \) and \( \xi = d^c f \) (notice that \( \delta d^c f = 0 \) by Kähler identities, or by observing that \( d^c f = -\delta(f \omega) \)). □

**Remark 1.23.3.** The operator \( f \mapsto \mathbb{L} = \delta \delta D^{-} df \) is a self-adjoint, semi-positive differential operator of order 4, acting on (real) functions. It can also be written as

\[ \mathbb{L}(f) = \frac{1}{2} \Delta^2 f - \delta(r(df)). \]

In this form, it has been first introduced by A. Lichnerowicz, see \[135\], and we shall then refer to it as the **Lichnerowicz fourth order operator** or, simply, the **Lichnerowicz operator**. It will play a prominent role in many parts of these notes. Notice that \( \mathbb{L} \) can be also written as

\[ \mathbb{L}(f) = (D^- d)^* D^- df, \]

where \((D^- d)^*\) denotes the formal adjoint of \( D^- d \). In particular, when \( M \) is compact, the kernel of \( \mathbb{L} \) is the space of (real) functions such that \( D^- df = 0 \), which, by Lemma 1.23.2, is the space of (real) functions \( f \) such that the gradient \( \text{grad}_g f \) is (real) holomorphic, also called **Killing potentials**, cf. Section 2.6.
CHAPTER 2

Holomorphic vector fields on a compact Kähler manifold

In this chapter, we present some well-known properties of the Lie algebra \( \mathfrak{h} \) of (real) holomorphic vector fields on a complex manifold \((M,J)\), as defined in Section 1.23, in the case when \((M,J)\) is a compact, kählerian manifold.

Most of these properties rely on the simple Lemma 2.1 in Section 2.1.

2.1. The space of holomorphic vector fields

**Lemma 2.1.** For any compact Kähler manifold \((M,g,J,\omega)\), any (real) holomorphic vector field can be uniquely written as

\[
X = X_H + \text{grad} f + J \text{grad} h,
\]

where \(X_H\) denotes the riemannian dual of a harmonic 1-form and where \(f\) and \(h\) are real functions, uniquely defined up to additive constants.

**Proof.** Let \(\xi = X^\flat\) be the dual 1-form of \(X\) with respect to the riemannian metric \(g\). From Lemma 1.23.2, we know that \(X\) is a real holomorphic vector field if and only if \(\xi\) satisfies \(D^- \xi = 0\). It follows that \(d\xi\) is \(J\)-invariant, so that \(dd^c \xi = 0\). By Lemma 1.17.1, the Hodge decomposition of \(\xi\) then reads:

\[
\xi = \xi_H + df + d^c h,
\]

where \(\xi_H\) is harmonic and \(f, h\) are real functions, uniquely defined up to additive constants. □

The real functions \(f\) and \(h\) appearing in (2.1.2), as well as the resulting complex function \(F := f + ih\), will be normalized by

\[
\int_M fv_g = \int_M hv_g = \int_M F v_g = 0,
\]

and thus uniquely defined.

**Definition 2.** For any (real) holomorphic vector field \(X\) in \(\mathfrak{h}\), the real function \(f\) appearing in (2.1.1) and normalized by (2.1.3) is called the real potential of \(X\) with respect to \(g\). It will be denoted \(f_X^g\), or simply \(f^X\) if the metric \(g\) is understood. The complex function \(F = f + ih\), normalized by (2.1.3), is called the complex potential of \(X\) with respect to \(g\) ad will be similarly denoted \(F_X^g\) or \(F^X\). A complex function \(F\) defined on \((M,J)\) is called a holomorphic potential with respect to some Kähler metric \(g\) if, after normalization, it is the complex potential with respect to \(g\) of some (real) holomorphic vector field.
Notice that for any $X$ in $\mathfrak{h}$, the real and complex potentials of $X$ and $JX$ are related by
\[(2.1.4)\quad f_{JX} = -h^X, \quad F_{JX} = iF^X.\]
This is because the space of $g$-harmonic 1-forms is $J$-invariant on any compact Kähler manifold, cf. Section 2.3 below. By (2.1.2), the Hodge decomposition of $J\xi$, is then $J\xi = J\xi_H - dh + d^c f$, with $J\xi_H = (J\xi)_H$.

**Remark 2.1.1.** For any fixed (real) holomorphic vector field $X$ and for any chosen Kähler class $\Omega$, the assignment: $g \rightarrow f^X_g$ can be regarded as a vector field on the space $M_{\Omega}$ of Kähler structures on $(M, J)$, with $[\omega] = \Omega$, cf. Section 3.1 below.

**Lemma 2.1.2.** For any (real) holomorphic vector field $X$ on a compact Kähler manifold $(M, g, J, \omega)$, the real potential $f^X_g$ of $X$ with respect to $g$ is determined by
\[(2.1.5)\quad \mathcal{L}_X\omega = df^X_g\]
and the normalization (2.1.3). In particular, $f^X_g = 0$ if and only if $X$ is Killing with respect to $g$, i.e. $\mathcal{L}_X g = 0$. Moreover:
\[(2.1.6)\quad f^X_g[X,Y] = \mathcal{L}_X f^Y_g - \mathcal{L}_Y f^X_g\]
for any two (real) holomorphic vector fields $X, Y$.

**Proof.** By using the Cartan formula, cf. Section 1.4, we get $\mathcal{L}_X\omega = d(X_\omega) = dJ\xi = dd^c f^X_g$. Moreover, since $M$ is compact, for any function $f$, $dd^c f = 0$ if and only if $f$ is constant, hence identically 0 when $f$ is normalized by (2.1.3); it follows that $f^X_g$ is entirely determined by (2.1.5). In particular, $f^X_g = 0$ if and only if $\mathcal{L}_X \omega = 0$; this, in turn, occurs if and only if $\mathcal{L}_X g = 0$. The identity (2.1.6) follows from
\[(2.1.7)\quad \int_M (\mathcal{L}_X f^Y_g - \mathcal{L}_Y f^X_g) v_g = \langle df^X_g, X_\omega \rangle + \langle df^Y_g, Y_\omega \rangle + \langle df^X_g, d^c h^X \rangle - \langle df^Y_g, d^c h^Y \rangle = 0.
\]

**Remark 2.1.2.** We check in a similar way that, for any $X$ in $\mathfrak{h}$,
\[(2.1.9)\quad \mathcal{L}_{X^{1,0}}\omega = \frac{1}{2} dd^c F^X_g,
\]
where $X^{1,0} = \frac{1}{2}(X - iJX)$ denotes the part of type $(1,0)$ of $X$ (see Section 1.8) and that
\[(2.1.10)\quad F^X_g[X,Y] = \mathcal{L}_{X^{1,0}} f^Y_g - \mathcal{L}_{Y^{1,0}} f^X_g.
\]
On any complex manifold \((M, J)\), the (real) Lie algebra \(\mathfrak{h}\) of (real) holomorphic vector fields can be viewed as a complex Lie algebra via the action of \(J\). Via the identification \(X \mapsto X^{1,0} = \frac{1}{2}(X - iJX)\) of \((TM, J)\) with \(T^{1,0}M\), \(\mathfrak{h}\), as a complex Lie algebra, is then identified with the space of holomorphic sections of \(T^{1,0}M\) with respect to the natural holomorphic structure of \(T^{1,0}M\), cf. Section 1.8. If \(M\) is compact, we then have:

**Proposition 2.1.1.** If \((M, J)\) is compact, \(\mathfrak{h}\), as a complex Lie algebra, is finite-dimensional and is the Lie algebra of the group, \(\text{Aut}(M, J)\), of automorphisms of \((M, J)\), which is then a finite-dimensional complex Lie group.

**Proof.** As just recalled, \(\mathfrak{h}\) is naturally identified with the space of holomorphic sections of \(T^{1,0}M\), equipped with its natural holomorphic structure \(\bar{\partial}\) defined in Section 1.8. For any auxiliary \(J\)-compatible riemannian metric \(g\) on \(M\), the operator \(\bar{\partial} + \bar{\partial}^*\), where \(\bar{\partial}^*\) denotes the adjoint of \(\bar{\partial}\) with respect to \(g\), as well as the corresponding Dolbeault laplacien \(\Box = (\bar{\partial} + \bar{\partial}^*)^2 = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}\), are elliptic operators, of order 1 and 2 respectively, and \(\mathfrak{h}\) can equivalently be regarded as the kernel of either of them. It is then a well-known general fact that the kernel of an elliptic differential operator acting on the sections of (real or complex) vector bundle over a compact manifold is finite dimensional. We then deduce the second part of the proposition from the general Theorem 3.1 of Chapter I in S. Kobayashi’s book [120], by observing that \(\mathfrak{h}\) clearly coincide with the space of (real) vector fields which generate global 1-parameter groups in \(\text{Aut}(M, J)\), as \(M\) is compact.

**Remark 2.1.3.** An alternative argument for Proposition 2.1.1, also to be found in Kobayashi’s book [120], relies on the fact that any almost-complex structure on a manifold \(M\) of dimension \(n = 2m\) can be regarded as a \(Gl(m, \mathbb{C})\)-structure on \(M\), where the complex linear group \(Gl(m, \mathbb{C})\) is viewed as as a subgroup of \(Gl(2m, \mathbb{R})\), and on the observation that the Lie algebra \(gl(m, \mathbb{C})\) of \(Gl(m, \mathbb{C})\) is elliptic, as a subspace of \(gl(2m, \mathbb{R})\), meaning that it contains no matrix of rank 1. According to the general Theorem 4.1 in [120, Chapter I]), the group of automorphism of any compact almost-complex manifold \((M, J)\) is then a (finite-dimensional, real) Lie group, whose Lie algebra is clearly the (real) Lie algebra of (real) vector fields \(X\) on \(M\) such that \(L_X J = 0\), which is then finite-dimensional. If \(J\) is integrable, this is a complex Lie algebra, and \(\text{Aut}(M, J)\) is then a complex Lie group, cf. Theorem 1.1 in [120, Chapter III].

**Remark 2.1.4.** In Proposition 2.1.1, the compactness assumption is crucial. If \((M, J)\) is not compact, it may happen that \(\text{Aut}(M, J)\) be infinite dimensional, e.g. \(\text{Aut}(\mathbb{C}^2)\) contains all transformations of the form \((z_1, z_2) \mapsto (z_1, z_2 + F(z_2))\), for any entire holomorphic function \(F\) on \(\mathbb{C}\). It may also happen that \(\text{Aut}(M, J)\) be a (finite-dimensional, real) Lie group, but not a complex Lie group, e.g. \(\text{Aut}(D) = PSL(2, \mathbb{R}) = Sl(2, \mathbb{R})/\pm 1\), if \(D\) denotes the open unit disk in \(\mathbb{C}\). In this exemple, \(\mathfrak{h}\) is still a complex Lie algebra but does not coincide with the Lie algebra of \(\text{Aut}(D)\), due to the fact

\footnote{Such a metric always exists, e.g. for any riemannian metric \(\tilde{g}\) on \(M\), \(g(X, Y) := \frac{1}{2}(\tilde{g}(X, Y) + \tilde{g}(JX, JY))\) is \(J\)-compatible.}
that not all (real) vector fields in $\mathfrak{h}$ are complete (compare the generator of the rotation group centered at the origin, which is evidently complete, and its image by $J$, which is evidently not complete). More generally, the group of automorphisms any bounded domain in $\mathbb{C}^{m+1}$ is a real, but not a complex, Lie group, cf. e. g. [155].

**Notation** 1. In the sequel the full group of automorphisms of a compact complex manifold $(M, J)$ will be denoted by $\text{Aut}(M, J)$, or simply $\text{Aut}(M)$ if $J$ is understood, whereas the identity component of $\text{Aut}(M, J)$, simply called the connected group of automorphisms, will be denoted by $\text{H}(M, J)$ — or $\text{H}(M)$.

### 2.2. The space of Killing vector fields

On a general riemannian manifold $(M, g)$, a (real) vector field $X$ is called a Killing vector field if it preserves $g$, i.e. if

\[
\mathcal{L}_X g = 0.
\]

Equivalently, $X$ is Killing if and only if $DX$ is $g$-skew-symmetric.

For any riemannian manifold $(M, g)$, the space of Killing vector field is a (real) Lie algebra which we shall denote by $\mathfrak{k}(M, g)$ or, simply, by $\mathfrak{k}$. If $(M, g)$ is complete, any Killing vector field is complete, cf. e. g. [121], and $\mathfrak{k}$ is therefore the Lie algebra of the Lie group $\text{Isom}(M, g)$ of automorphisms of $(M, g)$. If moreover, $M$ is compact, $\text{Isom}(M, g)$ is also compact.

**Notation** 2. The identity component of $\text{Isom}(M, g)$, simply called the connected isometry group of $(M, g)$, will be denoted by $K(M, g)$, or $K(M)$ when the metric $g$ is understood.

We then have the following general lemma:

**Lemma 2.2.1.** Let $M$ be any compact riemannian manifold. Let $\gamma$ be any element of $K(M, g)$ and $\psi$ any harmonic $p$-form. Then:

\[
\gamma^* \psi = \psi.
\]

In particular, for any $X$ in $\mathfrak{k}$, we have that

\[
\mathcal{L}_X \psi = 0.
\]

**Proof.** Since $\gamma$ belongs to $\text{Isom}_0(M, g)$ it induces the identity on all de Rham spaces $H^p(M, \mathbb{R})$. It follows that $\gamma^* \psi$ belongs to the same de Rham class as $\psi$. On the other hand, $\gamma^* \psi$ is certainly harmonic, as $\psi$ is an isometry. It follows that $\gamma^* \psi = \psi$. Let $X$ be any Killing vector field and denote by $\Phi^X_t$ the flow of $X$; then, $\Phi^X_t$ belongs to $K(M, g)$ for all values of $t$ so that $\mathcal{L}_X \psi := \frac{d}{dt}|_{t=0} \Phi^X_t \gamma^* \psi = 0$. Alternatively, by Cartan formula, $\mathcal{L}_X \psi = d(X \lrcorner \psi) + X \lrcorner d\psi = d(X \lrcorner \psi)$, as $\psi$ is closed; but, on a compact riemannian manifold, an exact harmonic $p$-form is identically equal to 0. □

As a direct corollary, we get

**Proposition 2.2.1.** On any compact Kähler manifold, the space of Killing vector fields is a Lie subalgebra of the Lie algebra of (real) holomorphic vector fields:

\[
\mathfrak{k} \subset \mathfrak{h}.
\]
More precisely, \( \mathfrak{k} \) is the subspace of divergence-free elements of \( \mathfrak{h} \).

**Proof.** Let \( X \) a Killing vector field. By Lemma 2.2.1, \( X \) preserves \( \omega \) (which is \( D \)-parallel, hence harmonic). It then preserves \( J \), hence belongs to \( \mathfrak{h} \). Any Killing vector field is divergence-free. Conversely, according to (2.1.2) we have \( \delta_g X^\flat = \Delta g X^\flat \), for any \( X \) in \( \mathfrak{h} \), so that \( X \) is divergence-free if and only if its real potential \( f^X_g \) is identically zero; according to Lemma 2.1.2, this occurs if and only if \( X \) is Killing with respect to \( g \). Alternatively, \( X \) is divergence-free if and only \( L_X v_g = L_X \omega = 0 \), whereas, by (2.1.5) and (1.12.2)–(1.15.6), \( L_X \omega^m = m \omega^{m-1} \wedge L_X \omega = -\Delta_g f^X_g \omega^m \), so that \( L_X v_g = 0 \) if and only \( \Delta_g f^X_g = 0 \), if and only if \( f^X_g = 0 \). □

### 2.3. Harmonic 1-forms and parallel vector fields

We start this section by recalling some basic facts concerning the space, \( \mathfrak{h} \), of (real) harmonic 1-forms on a compact Kähler manifold (recall that, in general, a \( p \)-form \( \phi \) defined on a Riemannian manifold \( (M, g) \) is called harmonic if \( \Delta \phi = 0 \), where \( \Delta \) denotes the Laplacian relative to \( g \)).

**Proposition 2.3.1.** On any compact Kähler manifold \( (M, g, J, \omega) \), a real 1-form \( \alpha \) is harmonic if and only if it satisfies the following two conditions:

\[
(2.3.1) \quad d \alpha = 0, \quad d^c \alpha = 0.
\]

Equivalently, \( \alpha \) is harmonic if and only if \( \alpha^{1,0} := \frac{1}{2}(\alpha + iJ\alpha) \) is holomorphic.

**Proof.** Since \( M \) is compact, a 1-form \( \alpha \) is harmonic if and only if \( d \alpha = 0 \) and \( \delta \alpha = 0 \). Since the metric is Kähler, we have \( \Delta = \Delta^c \), cf. Proposition 1.15.1, so that a harmonic 1-form \( \alpha \) is also \( d^c \)-closed, hence satisfies (2.3.1). Conversely, for any real 1-form \( \alpha \) satisfying (2.3.1), we have \( d \alpha = 0 \) and, by using (1.14.1), \( \delta \alpha = \Delta d^c \alpha = 0 \), hence \( \alpha \) is harmonic. From

\[
(2.3.2) \quad \partial \alpha^{1,0} = \frac{1}{4}(d - id^c)(\alpha + iJ\alpha) = \frac{1}{4}(d\alpha + Jd\alpha) + \frac{i}{4}(dJ\alpha + JdJ\alpha),
\]

we infer that \( \alpha^{1,0} \) is homomorphic whenever (2.3.1) is satisfied. Conversely, if \( \partial \alpha^{1,0} = 0 \), \( \alpha^{1,0} \) is \( \Box \)-harmonic, as it is evidently \( \partial^* \)-closed for type reason; by Proposition 1.15.1, it is then harmonic, as well as its real part \( \frac{1}{2} \alpha \) and its imaginary part \( \frac{1}{2} J\alpha \). □

As a direct corollary of Proposition 2.3.1, we get

**Proposition 2.3.2.** Let \( (M, J) \) be any compact complex manifold, and \( g \) any riemannian metric on \( M \) which is Kähler with respect of \( J \). Then, \( \mathfrak{h} \) is independent of \( g \) and is closed under the action of \( J \). Moreover, for any element \( \gamma \) of \( \mathcal{H}(M, J) \) and any element \( \alpha \) of \( \mathfrak{h} \), we have that

\[
(2.3.3) \quad \gamma^* \alpha = \alpha.
\]

**Proof.** The first part of the proposition readily follows from Proposition 2.3.1 since (2.3.1) as well as (2.3.2) are clearly independent of the Kähler metric \( g \). The last statement is quite reminiscent of Lemma 2.2.1 and is proved similarly. Any element \( \gamma \) of \( \mathcal{H}(M, J) \) is homotopic to the identity, hence induces the identity on \( H^1_{dR}(M, \mathbb{R}) \). It follows that \( \gamma^* \alpha \) still
belongs to the same de Rham class as $\alpha$. On the other hand, $\gamma^*\alpha$ still belongs to $\mathfrak{ham}$, as this space only depends on $J$, and is then harmonic; we then have $\gamma^*\alpha = \alpha$.

**Remark 2.3.1.** It readily follows from 1.23.9 in Lemma 1.23.10 that, on any compact Kähler manifold $(M, g, J, \omega)$, a (real) 1-form $\alpha$ is harmonic with respect to $g$ if and only if

$$D^\perp \alpha = 0,$$

i.e. if and only if $D\alpha$ is $J$-anti-invariant. In view of Proposition 2.3.2, this condition is actually independent of the chosen Kähler metric $g$ on $(M, J)$.

**Proposition 2.3.3.** An element $X$ of $\mathfrak{h}$ is $D$-parallel if and only if its riemannian dual 1-form $\xi = X^\flat$ is harmonic.

**Proof.** By remark 2.3.1, if $\xi$ is harmonic, $D\xi$ is $J$-anti-invariant, whereas, by Lemma 1.23.2, $D\xi$ is $J$-invariant, as $X$ belongs to $\mathfrak{h}$; it follows that $\xi$, hence also $X$, are parallel. Conversely, by (1.23.2), any parallel vector field $X$ belongs to $\mathfrak{h}$, and its dual 1-form $\xi$ is parallel, hence harmonic. □

The space of (real) parallel vector fields is denoted by $a$. We just saw that $a$ is a subspace of $\mathfrak{h}$. More precisely, we have

**Proposition 2.3.4.** The space $a$ of (real) parallel vector fields is a (complex) abelian Lie subalgebra of $\mathfrak{h}$ and is contained in the centre of $\mathfrak{h}$.

**Proof.** $a$ is clearly abelian, as $[X, Y] = D_X Y - D_Y X = 0$, if $X, Y$ both belong to $a$, and is clearly contained in $\mathfrak{h}$, because of lemma 1.23.1. We now show that $[Z, X] = 0$ for any $Z$ in $a$ and any $X$ in $\mathfrak{h}$. If $Z$ is parallel, so is $JZ$ and both vector fields $Z, JZ$ generate a foliation, say $\mathcal{F}$, whose each leaf is a 2-dimensional, totally geodesic, flat, complete Kähler submanifold of $M$ (the latter comes from the fact that all geodesics of the leaf are integral curves of constant linear combinations of $Z$ and $JZ$). It follows that the universal covering of each leaf of $\mathcal{F}$ is isometric to $\mathbb{C}$. Actually, the de Rham decomposition of the universal cover $\tilde{M}$ of $M$ is a Kähler product $\mathbb{C} \times N$, for some simply-connected Kähler manifold $N$, and, for each $y$ in $N$, $\mathbb{C} \times \{y\}$ is the universal cover of the corresponding leaf. Now, $X$ is of the form $X = X_H + \text{grad} f + J\text{grad} h$, where $X_H$ is the dual of the harmonic 1-form $\xi_H$. We know from Lemma 2.2.1 that $[Z, X_H] = (\mathcal{L}_Z \xi_H)^2 = 0$, as $Z$ is Killing. On the other hand, since $Z$ preserves $g$ and $J$, we then have $[Z, X] = \text{grad} \mathcal{L}_Z f + J\text{grad} \mathcal{L}_Z h$. By substituting $JZ$ to $Z$, we get $[JZ, X] = J[Z, X] = \text{grad} \mathcal{L}_{JZ} f + J\text{grad} \mathcal{L}_{JZ} h = -\text{grad} \mathcal{L}_Z h + J\text{grad} \mathcal{L}_Z f$. We infer $\mathcal{L}_Z f - \mathcal{L}_{JZ} h = 0$ and $\mathcal{L}_{JZ} f + \mathcal{L}_Z h = 0$ (recall that all these functions are of zero integral, as $Z$ and $JZ$ are divergence-free). This means that the complex potential $F := f + i h$ is holomorphic on each leaf of $\mathcal{F}$, hence determines a holomorphic function on its universal cover. On the other hand, $F$ is a globally defined smooth function on $M$, hence bounded. It follows that $F$ is constant on each leaf, as well as $f$ and $h$. From $[Z, X] = \text{grad} \mathcal{L}_Z f + J\text{grad} \mathcal{L}_Z h$ we then infer $[X, Z] = 0$. □


2.4. The ideal $\mathfrak{h}_{\text{red}}$ and the reduced automorphism group

For any (real) holomorphic vector field $X$ and for any harmonic 1-form $\alpha$, the scalar $\alpha(X)$ is the real part of $\alpha^{1,0}(X^{1,0})$, which is a holomorphic function, hence constant as $M$ is compact.

Equivalently, for any $X$ in $\mathfrak{h}$ and any $\alpha$ in $\text{harm}$ we have that

\[ \mathcal{L}_X \alpha = 0, \]

which readily follows from Proposition 2.3.2 (we can also argue as in Section 2.2: $\alpha$ is harmonic if and only if it is the real part of a holomorphic $(1,0)$-form; for any $X$ in $\mathfrak{h}$, $\mathcal{L}_X \alpha = d(\alpha(X))$ is then harmonic and exact, hence identically zero).

We thus get a $\mathbb{R}$-linear map

\[ \tau : \mathfrak{h} \to \text{harm}^*, \]

from $\mathfrak{h}$ to the (real) dual of $\text{harm}$: to each $X$ in $\mathfrak{h}$ we associate the linear form $\alpha \mapsto \alpha(X)$.

Moreover, for any two elements $X, Y$ of $\mathfrak{h}$ we have that $\tau([X,Y]) = 0$. Indeed, for each harmonic 1-form $\alpha$ and for any two elements $X, Y$ of $\mathfrak{h}$, we have

\[ \alpha([X,Y]) = -d\alpha(X,Y) + X \cdot \alpha(Y) - Y \cdot \alpha(X) = X \cdot \alpha(Y) - Y \cdot \alpha(X) = 0. \]

This means that the map $\tau$ is a Lie algebra morphism from $\mathfrak{h}$ to $\text{harm}^*$, when the latter is equipped with the trivial (abelian) Lie algebra structure.

The kernel of $\tau$ will be denoted by $\mathfrak{h}_{\text{red}}$: $\mathfrak{h}_{\text{red}}$ is then an ideal of $\mathfrak{h}$ and actually satisfies the stronger condition

\[ [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}_{\text{red}}. \]

An element $X$ of $\mathfrak{h}$ belongs to $\mathfrak{h}_{\text{red}}$ if and only if its dual $\xi$ is of the form $\xi = df + d^c h$, i.e. if and only if $\xi_H = 0$: this readily follows from

\[ \alpha(X) = \frac{1}{V} \langle \alpha, \xi \rangle = \frac{1}{V} \langle \alpha, \xi_H \rangle, \]

for any harmonic 1-form $\alpha$ (here, and henceforth, $V = \int_M v_g$ denotes the total volume of the metric).

The inclusion $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}_{\text{red}}$ can be made more precise via the following

**Proposition 2.4.1.** For any two vector fields $X = X_H + \text{grad} f^X + J \text{grad} h^X$ and $Y = Y_H + \text{grad} f^Y + J \text{grad} h^Y$ in $\mathfrak{h}$, the bracket $[X,Y]$ belongs to $\mathfrak{h}_{\text{red}}$, hence is of the form $[X,Y] = \text{grad} f^{[X,Y]} + J \text{grad} h^{[X,Y]}$, with

\[ f^{[X,Y]} = df^Y(X) - df^X(Y), \]

\[ h^{[X,Y]} = \omega(X,Y) - \frac{1}{V} \langle JX_H, Y_H \rangle + df^X(JY) - df^Y(JX). \]

In particular, if $X = \text{grad} f^X + J \text{grad} h^X$ and $Y = \text{grad} f^Y + J \text{grad} h^Y$ both belong to $\mathfrak{h}_{\text{red}}$, then

\[ f^{[X,Y]} = \{f^X, h^Y\} + \{h^X, f^Y\}, \]

\[ h^{[X,Y]} = \{h^X, h^Y\} - \{f^X, f^Y\}, \]
where, we recall, \(\{\cdot, \cdot\}\) denotes the Poisson bracket with respect to the Kähler form \(\omega\).

**Proof.** By using the computation already done in the proof of Proposition 1.5.1, as well as the identity (2.1.5), which determines the real potential of any element of \(h\), we get

\[
\iota_{[X,Y]}\omega = -d(\omega(X,Y)) + \iota_X dd^c f^Y - \iota_Y dd^c f^X \\
= -d(\omega(X,Y)) + \mathcal{L}_X d^c f^Y - \mathcal{L}_Y d^c f^X \\
- d(\iota_X d^c f^Y) + d(\iota_Y d^c f^X) \\
= -d(\omega(X,Y)) + \mathcal{L}_X d^c f^Y - \mathcal{L}_Y d^c f^X \\
- d(\iota_X d^c f^Y) + (\iota_Y d^c f^X) \\
= -d\mathcal{L}|X,Y| + d^c f^{[X,Y]}(\text{note that } \mathcal{L}_X \text{ and } \mathcal{L}_Y \text{ commute to } d^c, \text{ as } X, Y \text{ belong to } h). \]

This gives (2.4.4) (the constant \(-\frac{1}{\pi}\langle JX_H, Y_H \rangle\) — where, we recall, \(X_H, Y_H\) denote the riemannian duals of the harmonic parts of \(X^h, Y^h\) — has been added to guarantee that \(\int_M h^{[X,Y]}|_U = 0\)). Then, (2.4.5) follows easily, by using the expressions of the Poisson bracket given in Section 1.2. \(\square\)

It turns out that the map \(\tau\) can be integrated to a Lie group homomorphism \(\tilde{\tau}\) from \(H(M)\) to the *Albanese torus* \(\text{Alb}(M) := \text{harm}^*/\Gamma\), where \(\Gamma\) denotes the natural image of \(H_1(M, \mathbb{Z})\) in \(\text{harm}^*\), via the map \(\alpha \mapsto \int_0^\gamma \alpha\), for any loop \(c\) and any harmonic 1-form \(\alpha\). To explain this fact, we first fix a point \(x_0\) of \(M\) and we consider the *Jacobi map*, \(A\), from \(M\) to \(\text{Alb}(M)\) defined as follows [32]: to each \(x\) we associate \(A(x) : \alpha \mapsto \int_{x_0}^x \alpha\), for each harmonic 1-form \(\alpha\), where the integral runs over some path form \(x_0\) to \(x\); any two such paths differ by a loop, so that \(A(x)\) actually lives in \(\text{Alb}(M)\). For any \(\gamma\) in \(H(M)\), we have that

\[
A(\gamma \cdot x)(\alpha) = \int_{x_0}^{\gamma \cdot x_0} \alpha = \int_{x_0}^{\gamma \cdot x_0} \alpha + \int_{x_0}^{\gamma \cdot x_0} \alpha \mod \Gamma \\
= \int_{x_0}^{\gamma \cdot x_0} \gamma^* \alpha + \int_{x_0}^{\gamma \cdot x_0} \alpha \mod \Gamma \\
= \int_{x_0}^{\gamma \cdot x_0} \alpha + \int_{x_0}^{\gamma \cdot x_0} \alpha \mod \Gamma \\
= A(x)(\alpha) + \int_{x_0}^{\gamma \cdot x_0} \alpha \mod \Gamma 
\]

(recall that \(\gamma^*(\alpha) = \alpha\), cf. Proposition 2.3.2). In short, we have

(2.4.6) \(A(\gamma \cdot x) = A(x) + \tilde{\tau}(\gamma) \mod \Gamma\)

with

(2.4.7) \(\tilde{\tau}(\gamma) = \{\alpha \mapsto \int_{x_0}^{\gamma \cdot x_0} \alpha\} \mod \Gamma\)
for each $\alpha$ in $\mathfrak{h}_{\text{harm}}$. Then, $\tilde{\tau}$ is a homomorphism from $H(M,J)$ to $\text{Alb}(M)$. This homomorphism is actually independent of the choice of $x_0$. Indeed, if $x_1$ is another base-point, we have

\[
\int_{x_1}^{\gamma x_1} \alpha = \int_{x_1}^{x_0} \alpha + \int_{x_0}^{\gamma x_0} \alpha + \int_{\gamma x_0}^{\gamma x_1} \alpha \mod \Gamma \\
= \int_{x_0}^{\gamma x_0} \alpha + \int_{x_0}^{x_1} \alpha + \int_{x_0}^{x_0} \gamma^* \alpha \mod \Gamma \\
= \int_{x_0}^{\gamma x_0} \alpha \mod \Gamma,
\]

for any $\alpha$ in $\mathfrak{h}_{\text{harm}}$ and any $\gamma$ in $H(M)$, again because $\gamma^* \alpha = \alpha$. Moreover, the derivative of $\tilde{\tau}$ at the identity clearly coincides with $\tau$. It follows that the ideal $\mathfrak{h}_{\text{red}}$ is the Lie algebra of a (closed) connected Lie subgroup, say $H_{\text{red}}(M,J)$, of $H(M,J)$, namely the identity component of the kernel of $\tilde{\tau}$ in $H(M,J)$.

**Definition 3.** The group $H_{\text{red}}(M,J)$ — or simply $H_{\text{red}}(M)$ if $J$ is understood — will be referred to as the reduced automorphism group of $(M,J)$.

**Remark 2.4.1.** It can be shown that the kernel of $\tilde{\tau}$ in $H(M,J)$ has a finite number of connected components and that the reduced automorphism group $H_{\text{red}}(M,J)$ is a linear algebraic group, [82, Theorem 5.5], [137, Theorem 3.12].

We end this section with the following alternative characterization of $\mathfrak{h}_{\text{red}}$:

**Proposition 2.4.2.** For any compact Kähler manifold $(M,g,J,\omega)$, the ideal $\mathfrak{h}_{\text{red}}$ coincides with the space of (real) holomorphic vector fields whose zero set is non-empty.

**Proof.** This theorem was first established by Y. Matsushima [149], [120, Theorems 9.7 and 9.8], in the case when $M$ is a projective manifold polarized by a holomorphic line bundle, cf. Section 8.1, then, in the general Kähler case, by J. B. Carrell and D. I. Lieberman in [50]. We here follow the nice elementary argument given by C. LeBrun-S. Simanca in [130].

Let $X$ a (real) holomorphic vector field on $M$, whose zero set is non-empty: for any (real) harmonic 1-form $\alpha$, $\alpha(X)$ is constant, hence equal to zero; then, $X$ belongs to $\mathfrak{h}_{\text{red}}$. Conversely, suppose that $X = \text{grad}_g f + J\text{grad}_g h$ belongs to $\mathfrak{h}_{\text{red}}$ and denote by $c \geq 0$ the minimum of $|X|^2$ on $M$; we have to prove that $c = 0$.

The following argument, taken from [130], holds for any real vector field $X$ of the form $X = \text{grad}_g f + J\text{grad}_g h$ with the only assumption that $X$ commutes with $JX = -\text{grad}_g h + J\text{grad}_g f$ (this assumption is certainly satisfied if $X$ belongs to $\mathfrak{h}_{\text{red}}$, as we then have $L_X J = 0$, so that $[X,JX] = J[X,X] = 0$); the flows, $\Phi^X_t$ and $\Phi^{JX}_t$, of $X$ and $JX$ — which are both globally defined as $M$ is compact — commute and thus determine a $\mathbb{C}$-action on $M$: for any point $x_0$ in $M$ — for convenience, we choose $x_0$ such that $|X(x_0)|^2 = c$ — the $\mathbb{C}$-orbit of $x_0$ is the image of the map $\Phi : \mathbb{C} \to M$ defined by $\Phi(z) = \Phi^X_s \circ \Phi^{JX}_t(x_0) = \Phi^{JX}_t \circ \Phi^X_s(x_0)$, for any $z = s + it$ in $\mathbb{C}$; we then define $H : \mathbb{C} \to \mathbb{C}$ by $H(z) = F(\Phi(z))$, with $F = f + ih$;
Then \((\partial H/\partial s)(z) = (X \cdot F)(\Phi(z))\) and \((\partial H/\partial t)(z) = (JX \cdot F)(\Phi(z))\). For any \(r \geq 0\), denote by \(D_r\) the (closed) disk of radius \(r\) in \(\mathbb{C}\). If \(r > 0\), denote by \(\partial D_r\) the boundary circle, of radius \(r\). Then, \(\left| \int_{\partial D_r} H \, dz \right|\) is clearly bounded from above by \(2\pi r C\), where \(C\) denotes the maximal value of \(|F|\) on \(M\). On the other hand, \(\int_{\partial D_r} H \, dz\) is also equal to \(\int_{D_r} \partial_H \wedge dz\), from \(X \cdot F = df(X) + i db(X) = |df|^2 + (df, df) + i (dh, df)\) and \(JX \cdot F = df(JX) + i db(JX) = -(df, dh) + i (dh, df)\). Hence, we readily get \(\partial_H \wedge dz = \frac{i}{2} \partial_H \wedge d\partial_H\) \(ds \wedge dt = i X(\Phi(z))^2\) \(ds \wedge dt\). We infer that \(\left| \int_{D_r} H \wedge dz \right|\) is bounded from below by \(\pi r^2 C\). By comparing the above upper and lower bounds for \(\int_{\partial D_r} H \, dz = \int_{D_r} \partial_H \wedge dz\), we eventually get: \(C \geq \frac{\pi}{2} r\), for any \(r > 0\); this, of course, is only possible if \(c = 0\), i.e. \(X(x_0) = 0\) (\(\Phi\) is then reduced to the constant map \(\Phi(z) = x_0\)).

**Remark 2.4.2.** The existence of (non-trivial) holomorphic vector fields with a non-empty zero set on a compact Kähler manifold has strong consequences concerning the structure of the underlying complex manifold. The interested reader will find a detailed account in S. Kobayashi's book \([120]\), cf. also \([50], [51], [137], [84]\) for further information.

2.5. The Calabi operator \(L^+\) and its conjugate

Any element of \(\mathfrak{h}_{\text{red}}\) is of the form \(X = \text{grad} f + J \text{grad} h\), where \(f, h\) are real functions with zero mean value. Conversely, by Lemma 1.23.2, such a (real) vector field belongs to \(\mathfrak{h}\), hence to \(\mathfrak{h}_{\text{red}}\), if and only if

\[
D^- (df + dh) = 0.
\]

In general, any \(J\)-anti-invariant bilinear form, \(B\), splits as \(B = B^{2,0} + B^{0,2}\), where the \((2,0)\)-part \(B^{2,0}\) is defined by \(B^{2,0}(X, Y) = \frac{1}{2} (B(X, Y) - i B(JX, Y))\), and the \((0, 2)\)-part \(B^{0,2}\) is defined by \(B^{0,2}(X, Y) = \frac{1}{2} (B(X, Y) + i B(JX, Y))\). More generally, for any bilinear form \(B, B^{2,0}\), resp. \(B^{0,2}\), will denote the \((2,0)\)-part, resp. the \((0,2)\)-part, of the \(J\)-anti-invariant part, \(B^-\), of \(B\). In the sequel, for any \(1\)-form \(\alpha\) — real or complex — we denote by \(D^{2,0} \alpha\) the \((2,0)\)-part of \(D^- \alpha\) and by \(D^{0,2} \alpha\) its \((0,2)\)-part. We then easily check:

\[
D^{2,0} \alpha = \frac{1}{2} (D^- \alpha + i D^- J \alpha) = D^- \alpha^{1,0} = D^1 \alpha^{1,0},
\]

\[
D^{0,2} \alpha = \frac{1}{2} (D^- \alpha - i D^- J \alpha) = D^- \alpha^{0,1} = D^0 \alpha^{0,1}.
\]

Notice in particular that:

\[
D^{2,0} J \alpha = -i D^{2,0} \alpha, \quad D^{0,2} J \alpha = i D^{0,2} \alpha.
\]

By introducing the complex potential \(F = f + i h\) of \(X\), cf. Definition 2, the condition (2.5.1) is then equivalent to

\[
D^{0,2} dF = 0.
\]

Indeed, since \(df + dh\) is real, the condition (2.5.1) is satisfied if and only if \(D^{2,0}(df + dh) = D^{0,2}(df + dh) = 0\), whereas, by (2.5.3), \(D^{0,2}(df + dh) = D^{0,2}dF\).
2.5. THE CALABI OPERATOR $L^+$ AND ITS CONJUGATE

We then define the \textit{Calabi operator} $\mathbb{L}^+$, acting on complex functions, by

\begin{equation}
(2.5.5)
L^+(F) = 2\bar{\delta}D^{0,2}dF = 2(D^{0,2}d)^*D^{0,2}dF,
\end{equation}

and the conjugate operator $\mathbb{L}^-$ by:

\begin{equation}
(2.5.6)
L^-(F) := L^+(\overline{F}) = 2\bar{\delta}D^{2,0}dF = 2(D^{2,0}d)^*D^{2,0}dF,
\end{equation}

where $(D^{2,0}d)^*$, resp. $(D^{0,2}d)^*$, denotes the adjoint operator of $D^{2,0}d$, resp. $D^{0,2}d$, with respect to the natural global hermitian inner products defined on the space of complex-valued functions and on the space of complex valued-bilinear forms of type $(2,0)$, resp. of type $(0,2)$. In particular,

\begin{equation}
(2.5.7)
\delta\delta(D^-(df + dh)) = \Re\mathfrak{Re}(L^+(F)),
\end{equation}

for any real functions $f, h$, with $F = f + ih$.

Both operators $L^+$ and $L^-$ are elliptic, self-adjoint, semi-positive fourth order differential operators. Moreover, it follows from (2.5.2) and from Lemma 1.23.5 that $L^+$ and $L^-$ are related to the fourth order Lichnerowicz operator $\mathbb{D} = \delta\delta D^\star$ defined by (1.23.14), cf. also Remark 1.23.3), by:

\begin{equation}
(2.5.8)
L^+(F) = \mathbb{L}(F) + i\mathbb{L}_\mathbb{K}F,
\end{equation}

and

\begin{equation}
(2.5.9)
L^-(F) = \mathbb{L}(F) - i\mathbb{L}_\mathbb{K}F,
\end{equation}

for any \textit{complex} function $F$, where, we recall, $\mathbb{K}$ denotes the hamiltonian vector field $\mathbb{K} = J\text{grad} s$ whose momentum is the scalar curvature (notice that the additional algebraic operators $\pm \frac{1}{2}\mathbb{L}_\mathbb{K}$ are self-adjoint, as the vector field $\mathbb{K}$, like any hamiltonian vector field, is divergence-free).

We then have (cf. [135, Section 95]):

\begin{proposition}
A (real) vector field $X$ belongs to $\mathfrak{h}_{\text{red}}$ if and only if it is of the form $X = \text{grad} f + J\text{grad} h$, where $f, h$ are real functions satisfying
\end{proposition}

\begin{equation}
(2.5.10)
L^+(F) = 0,
\end{equation}

with $F = f + ih$. Equivalently, the pair $(f, h)$ is a solution to the system:

\begin{equation}
(2.5.11)
\mathbb{L}(f) - \frac{1}{2}\mathbb{L}_\mathbb{K}h = 0,
\end{equation}

\begin{equation}
(2.5.12)
\mathbb{L}(h) + \frac{1}{2}\mathbb{L}_\mathbb{K}f = 0,
\end{equation}

where $\mathbb{K} := J\text{grad} s$.

\begin{proof}
We already observed that $X = \text{grad}_g f + J\text{grad}_g h$ belongs to $\mathfrak{h}_{\text{red}}$ if and only if $D^{0,2}dF = 0$, where $F = f + ih$. This, in turn, implies $L^+(F) = 0$. Conversely, $L^+(F) = 0$ implies $0 = \langle L^+(F), F \rangle = \int_M |D^{0,2}dF|^2 v_g$, hence $D^{0,2}dF = 0$. Since $\mathbb{L} = \delta\delta D^\star$ is a real operator, by (2.5.8), the condition $L^+(F) = 0$ is clearly equivalent to the system (2.5.11)-(2.5.12).
\end{proof}

\begin{remark}
For any (real) vector field $X$, the $(1,0)$-part $X^{1,0} = \frac{1}{2}(X - iJX)$ of $X$ is a holomorphic section of $T^{1,0}M$ — cf. Sections 1.8 and 2.1 — and the (complexified) riemannian dual of $X^{1,0}$ is the $(0,1)$-part,
\end{remark}
\( \xi^{(0,1)} \), of the riemannian dual, \( \xi = X^\flat \), of \( X \); we then have \( \xi^{(0,1)} = \frac{1}{2}(\xi - iJ\xi) \) and the Hodge decomposition (2.1.2) then implies:

\[
(2.5.13) \quad \xi^{(0,1)} = \xi_H^{(0,1)} + \bar{\partial}F,
\]

where \( F = f + ih \), \( \bar{\partial} = \frac{1}{2}(d - id^c) \) denotes the usual Cauchy-Riemann operator, acting on functions, and \( \xi_H^{(0,1)} \) is the \((0, 1)\) part of \( \xi_H \), which is also the \( \Delta\)-harmonic and the \( \Box\)-harmonic part of \( \xi^{(0,1)} \). In particular, \( \xi^{(0,1)} \) is \( \bar{\partial}\)-closed and (2.5.13) is the Dolbeault decomposition of \( \xi^{(0,1)} \), cf. the proof of Proposition 1.17.3. Then, the map

\[
(2.5.14) \quad F \mapsto (i\bar{\partial}F)^\sharp
\]
determines a \( \mathbb{C}\)-linear isomorphism from the space of complex-valued functions \( F \) which satisfy \( L^+(F) = 0 \) and \( \int_M F v_g = 0 \) to the space \( \mathfrak{h}_{\text{red}} \), which, by Proposition 2.4.1, is a (complex) Lie algebra isomorphism when the kernel of \( L^+ \) is equipped with the Poisson bracket (\( \mathbb{C}\)-linearly extended to the space of complex functions). Notice that this shows a posteriori that the kernel of \( L^+ \) is closed with respect to the Poisson bracket.

2.6. The space of hamiltonian Killing vector fields

The space \( \mathfrak{k}^{\text{ham}} := \mathfrak{h}_{\text{red}} \cap \mathfrak{K} \) is the space of hamiltonian Killing vector fields, i.e. the space of Killing vector fields of the form \( X = J\text{grad}_g h = \text{grad}_g \omega \). As a direct consequence of Proposition 2.5.1, we then have (see [135, Section 96]):

**Proposition 2.6.1.** A (real) holomorphic vector field \( X \) is a hamiltonian Killing vector field, i.e. belongs to \( \mathfrak{k}^{\text{ham}} \), if and only if \( X \) is of the form \( X = J\text{grad}_g h \), where \( h \) is a solution to the equation

\[
(2.6.1) \quad L(h) = \delta \delta D^\delta dh = 0.
\]

**Proof.** By definition, any vector field \( X \) in \( \mathfrak{k}^{\text{ham}} \) is of the form \( X = \text{grad}_\omega h = J\text{grad}_g h \) for some real function \( h \). By Proposition 2.5.1, such a vector field belongs to \( \mathfrak{h} \) — hence to \( \mathfrak{h}_{\text{red}} \), hence to \( \mathfrak{k}^{\text{ham}} \) — if and only if \( h \) is a solution to the system

\[
(2.6.2) \quad \delta \delta D^\delta dh = 0, \quad L_X h = 0.
\]

Now, the first equation \( \delta \delta D^\delta dh = 0 \) is equivalent to \( D^\delta dh = 0 \) — as \( M \) is compact — and, by Lemma 1.23.2, this is already equivalent to \( X \) being (real) holomorphic, hence Killing. It then remains to check that the second equation is redundant, i.e. a consequence of the first one: this is because \( L_X h = (d^c s, dh) = -(ds, d^c h) = -L_X s = 0 \), as \( X \) is a Killing vector field.

**Definition 4.** A real function \( h \) is called a Killing potential with respect to \( g \) if the hamiltonian vector field \( X := J\text{grad}_g h = \text{grad}_\omega h \) is a Killing vector field — equivalently, a (real) holomorphic vector field.

Notice that in Definition 4 we don’t demand that \( h \) be normalized by \( \int_M h v_g = 0 \), i.e. we don’t exclude constant real functions: these are then Killing potentials for any Kähler metric.
In view of Proposition 2.6.1, for any (compact, connected) Kähler manifold \((M, g, J, \omega)\) the space of Killing potentials with respect to \(g\) coincides with the kernel of the Lichnerowicz operator \(L\) in \(C^\infty(M, \mathbb{R})\).

As in Section 2.5, this shows a posteriori that the kernel of \(L\) is closed with respect to the Poisson bracket.

**Proposition 2.6.2 (A. Lichnerowicz [135]).** Let \((M, g, J, \omega)\) be a compact Kähler manifold. Assume that the Ricci tensor \(r_g\) is positive definite, with \(r_g \geq k g\), where \(k\) is a positive real number. Then,

\[
\lambda_1 \geq 2k,
\]

where \(\lambda_1\) denotes the smallest positive eigenvalue of the riemannian laplacian \(\Delta_g\) acting on functions. Moreover, \(\lambda_1 = 2k\) if and only if all eigenfunctions relative to \(\lambda_1\) are Killing potentials.

**Proof.** Let \(\lambda\) be any positive eigenvalue of \(\Delta_g\) and let \(f\) be any non-zero real function on \(M\) such that \(\Delta_g f = \lambda f\). Since \(\lambda > 0\), \(f\) is non-constant and \(\Delta_g df = d\Delta_g f = \lambda df\). By using (1.23.10), with \(\alpha = df\), by considering the inner product of both sides with \(df\) and by integrating over \(M\), we get

\[
\int_M |D^-df|^2 v_g = \frac{1}{2} \int_M |df|^2 v_g - \int_M r_g(\text{grad}_g f, \text{grad}_g f) v_g \leq \left( \frac{1}{4} - k \right) \int_M |df|^2 v_g.
\]

This readily implies (2.6.3). Moreover, \(\lambda = 2k\) if and only if \(f\) is a Killing potential.

Notice that for a general compact riemannian manifold \((M, g)\) with \(r_g \geq k g\), a similar argument using the Bochner formula (1.22.1) instead of (1.23.10) would imply \(\lambda_1 \geq k\), instead of the refined Kähler inequality (2.6.3).

**Remark 2.6.1.** For any real vector field \(X\) in \(\mathfrak{h}\), whose dual 1-form \(\xi\) with respect to any Kähler metric \(g\) is \(\xi_g = \xi_H + df_g^X + d^c h_g^X\), with the usual notation, we have

\[
\mathcal{L}(f_g^X) = \frac{1}{2} \left( ds_g, \xi_g - df_g^X \right) = \frac{1}{2} \delta \left( s_g (\xi_g - df_g^X) \right).
\]

Indeed, since \(X\) belongs to \(\mathfrak{h}\), we have \(\delta \delta D^- \xi_g = 0\), hence \(\mathcal{L}(f_g^X) = -\delta \delta D^- \xi_H - \delta \delta D^- d^c h_g^X\); from Lemma 1.23.4, the adjunction identity (1.23.16) and the Bianchi identity (1.19.9), we infer \(\delta \delta D^- \xi_H = \delta (J \xi_H \cdot \rho) = -(J \xi_H, \delta \rho) = \frac{1}{2} (\xi_H, ds_g)\), whereas, by (1.23.15), we have \(\delta \delta D^- d^c h = -\frac{1}{2} K h_g^X = \frac{1}{2} (ds_g, d^c h_g^X)\).

We thus get the first equality in (2.6.4); the second equality follows from \(\xi_g - df_g^X\) being co-closed. In particular, if the scalar curvature \(s_g\) is constant, then \(f_g^X\) is a Killing potential, as \(\xi_g - df_g^X\) is co-closed, and so is \(h_g^X\) as well; it follows that each component, \(\xi_H, df_g^X\) and \(d^c h_g^X\), of \(\xi_g\) is the dual of a real holomorphic vector field, \(X_H, \text{grad}_g f_g^X\) and \(J \text{grad}_g h_g^X\) respectively. We infer that \(J \text{grad}_g h_g^X = \text{grad}_g h_g^X\) belongs to \(\mathfrak{h}_{\text{ham}}\), that \(\text{grad}_g f_g^X = - J \text{grad}_g h_g^X\) belongs to \(J \mathfrak{h}_{\text{ham}}\) and that \(X_H\) is \(D^g\)-parallel, cf. Proposition 2.3.3. We thus get the decomposition (3.6.1) in Theorem 3.6.1 below.
CHAPTER 3

Calabi extremal Kähler metrics

3.1. The space of Kähler metrics within a fixed Kähler class

In this section, we fix a complex structure $J$ of $M$ and an element $\Omega$ of the de Rham space $H^2_{dR}(M, \mathbb{R})$ of $M$. We denote by $\mathcal{M}_\Omega$ the set of all Kähler metrics $(g, J, \omega)$ whose Kähler form $\omega$ belongs to $\Omega$.

Elements of $\mathcal{M}_\Omega$ will be indifferently designated by the metric $g$ or by the (symplectic) Kähler form $\omega$, or by the pair $(g, \omega)$. We assume that $\Omega$ is a Kähler class, i.e. that $\mathcal{M}_\Omega$ is non-empty. It is then a Fréchet manifold of infinite dimension.

More precisely, by the $dd^c$-lemma any two elements $\omega_0$ and $\omega$ are linked by

$$\omega = \omega_0 + dd^c \phi,$$

for some real function $\phi$ uniquely defined up to additive constant. In other words, for any choice of a base-point $\omega_0$, $\mathcal{M}_\Omega$ is identified with the subset of $[\phi]$ in $C^\infty(M, \mathbb{R})/\mathbb{R}$ such that $\omega_0 + dd^c \phi$ is positive definite. Here, $[\phi]$ denotes the class of $\phi$ modulo additive real constant.

For any $g$ in $\mathcal{M}_\Omega$, the tangent space $T_g \mathcal{M}_\Omega$ is then identified with the space, $A^{1,1}_{0,0}$, of exact $J$-invariant 2-forms. Via the operator $dd^c$, we eventually get the following identification:

$$T_g \mathcal{M}_\Omega = A^{1,1}_{0,0} = C^\infty(M, \mathbb{R})/\mathbb{R} = C^{\infty}_{0,g}(M, \mathbb{R}),$$

where $C^\infty(M, \mathbb{R})$ denotes the space of smooth real functions on $M$, $\mathbb{R}$ the space of (real) constant functions and $C^{\infty}_{0,g}(M, \mathbb{R})$ the space of real functions $f$ of $M$ such that

$$\int_M f \, v_g = 0$$

(more explanation about this normalization can be found in Section 3.1).

A vector field on $\mathcal{M}_\Omega$ can thus be viewed as an assignment $\mathcal{M}_\Omega \ni g \mapsto f_g \in C^{\infty}_{0,g}$.

The connected group of automorphisms $H(M, J)$ acts trivially on $H^2_{dR}M$, hence preserves $\Omega$ and thus acts on $\mathcal{M}_\Omega$. More generally, denote by $H_\Omega(M, J)$, or simply $H_\Omega$ if $J$ is understood, the subgroup of those elements of $\text{Aut}(M, J)$ which preserve $\Omega$. This group may not be connected but has at most a finite number of connected components, cf. [82, Theorem 4.8], [137, Proposition 2.2].

For convenience, we consider the right action of $H_\Omega(M, J)$ on $\mathcal{M}_\Omega$ defined by:

$$\gamma \cdot \omega = \gamma^* \omega = \gamma^{-1} \cdot \omega.$$
The infinitesimal action on $\mathcal{M}_\Omega$ of an element $X$ of $\mathfrak{h}$ is then the vector field $\hat{X}$ determined by

$$(3.1.5) \quad g \mapsto L_X \omega = d(X \cdot \omega) = dd^c f_g X.$$

Via the identification $T_g \mathcal{M}_\Omega = C^\infty_{0,g}$, $\hat{X}$ is then the vector field

$$(3.1.6) \quad \hat{X}_g = f_g X,$$

where $f_g X$ stands for the (real) potential of $X$ with respect to $g$ defined in Section 2.1.

### 3.2. The Calabi functional

The *Calabi functional*, $\mathcal{C}$, is the real function defined on $\mathcal{M}_\Omega$ by:

$$\mathcal{C}(g) = \int_M s^2_g v_g.$$

This function is clearly $H_\Omega(M,J)$-invariant:

$$\mathcal{C}(\gamma^* g) = \int_M s^2_g \gamma^* v_g = \int_M \gamma^*(s^2_g v_g) = \mathcal{C}(g),$$

for all $\gamma$ in $H_\Omega(M,J)$ and all $g$ in $\mathcal{M}_\Omega$.

**Definition 5.** A Kähler metric is called extremal — with respect to a Kähler class $\Omega$ — if it is a critical metric of the Calabi functional $\mathcal{C}$ in $\mathcal{M}_\Omega$.

As observed in the previous section, a vector of $\mathcal{M}_\Omega$ at $g$ is naturally identified with a (real) function $\phi$ of $M$ satisfying (3.1.3). The corresponding variation of $\omega$ is then:

$$\dot{\omega} = dd^c \dot{\phi}.$$

The corresponding variation of the various elements appearing in the definition of the Calabi functional $\mathcal{C}$ are then given by the following lemma:

**Lemma 3.2.1.** For any variation $\dot{\omega} = dd^c \dot{\phi}$ of $g$ in $\mathcal{M}_\Omega$, the first variation of the volume form $v_g$, of the Ricci form, of the scalar curvature $s_g = s$ and of $s v_g$ are given by:

- **varmu** (3.2.4)
$$\dot{v}_g = -\Delta \dot{\phi} v_g,$$

- **varricci** (3.2.5)
$$\dot{\rho} = \frac{1}{2} dd^c \Delta \dot{\phi},$$

- **vars** (3.2.6)
$$\dot{s} = -\Delta^2 \dot{\phi} - 2(dd^c \dot{\phi}, \rho) = -2\delta \delta D^- d\dot{\phi} + (ds, d\dot{\phi}),$$

and

- **varsmu** (3.2.7)
$$(s \dot{v}_g) = \delta(-2\delta D^- d\dot{\phi} - sd\dot{\phi}) v_g.$$

**Proof.** From (1.2.3) and (1.15.6) we infer: $\dot{v}_g = \dot{\omega} \wedge \omega^{m-1}_{[m-1]} = \Lambda(dd^c \dot{\phi}) v_g = -\Delta \dot{\phi} v_g$. The identity (3.2.5) follows from (1.19.6) and (3.2.4). The scalar curvature $s$ is deduced from the Ricci form $\rho$ by

$$\frac{1}{2m} s \omega^m = \rho \wedge \omega^{m-1},$$

and
or, equivalently, by
\begin{equation}
(3.2.9) \quad s = 2\Lambda \rho,
\end{equation}
where \( \Lambda \) is defined by (1.12.3) or, alternatively, cf. (1.12.2), by:
\begin{equation}
(3.2.10) \quad \Lambda(\psi) \omega^m = m \psi \wedge \omega^{m-1},
\end{equation}
for any 2-form \( \psi \). From (3.2.10), we infer: \( \dot{\Lambda}(\psi) \omega^m = -m \Lambda(\psi) \omega \wedge \omega^{m-1} + m(m-1) \psi \wedge \dot{\omega} \wedge \omega^{m-2} \). By using (1.12.2) and (1.12.5), we thus get:
\begin{equation}
(3.2.11) \quad \dot{\Lambda}(\psi) = -⟨\dot{\omega}, \psi⟩,
\end{equation}
for any variation \( \dot{\omega} \) of \( \omega \), hence
\begin{equation}
(3.2.12) \quad \dot{\Lambda}(\psi) = -(dd^c \dot{\phi}, \psi),
\end{equation}
in the current situation when \( \dot{\omega} = dd^c \dot{\phi} \). From (3.2.5) and (3.2.10), we then infer: \( \dot{s} = 2\dot{\Lambda}(\rho) + 2\Lambda(\rho) = -2(dd^c \dot{\phi}, \rho) - \Delta^2 \dot{\phi} \), which is the first expression of \( \dot{s} \) in (3.2.6), whereas the second expression is readily deduced from (1.23.14). Finally, (3.2.7) is a direct consequence of this expression and (3.2.4). □

Remark 3.2.1. In view of the Apte formulae — see Appendix A — any other functional on \( \mathcal{M}_Ω \) involving an invariant quadratic form in the curvature, like e.g. \( g \rightarrow \int_M |R|^2 v_g \), \( g \rightarrow \int_M |r|^2 v_g \) or \( g \rightarrow \int_M |W^K|^2 v_g \), is an affine function of \( C \), hence gives rise to the same critical metrics in \( \mathcal{M}_Ω \). Also notice that the total volume \( V_Ω := \int_M v_g \) and the total scalar curvature \( S_Ω := \int_M s_g v_g \) are constant as functions defined on \( \mathcal{M}_Ω \), hence only depend on the Kähler class \( Ω \) and on the complex structure \( J \); more precisely, from (1.2.3) and (3.2.8) we readily obtain
\begin{equation}
(3.2.13) \quad V_Ω = \left( \frac{Ω^m}{m!}, ([M]) \right),
\end{equation}
\begin{equation}
S_Ω = 4π \langle c_1(M, J) \cup \frac{Ω^{m-1}}{(m-1)!}, [M] \rangle.
\end{equation}
The fact that \( V_Ω \) and \( S_Ω \) are constant on \( \mathcal{M}_Ω \) can also be derived from (3.2.4) and (3.2.7), from which we readily infer \( \int_M v_g = 0 \) and \( \int_M (s_g v_g) = 0 \).

Theorem 3.2.1 (E. Calabi [45]). The first derivative, \( \dot{C} \), of the Calabi functional along any variation \( \dot{ω} = dd^c \dot{ϕ} \) of \( g \) in \( \mathcal{M}_Ω \) is given by
\begin{equation}
(3.2.14) \quad \dot{C} = -4\langle \dot{ϕ}, dδD^- ds \rangle = -4\langle D^- ds, D^- d\dot{ϕ} \rangle.
\end{equation}
A metric \( g \) in \( \mathcal{M}_Ω \) is then extremal if and only if the scalar curvature \( s_g \) is a Killing potential.

Proof. By using (3.2.6) and (3.2.4), as well as (1.23.14), we get
\begin{align*}
\dot{C} &= \int_M (2s\dot{s} - s^2 \Delta \dot{ϕ}) v_g \\
&= \int_M \left( -2s\Delta \dot{ϕ} - 4s(dd^c \dot{ϕ}, ρ) - 2s (d\dot{ϕ}, ds) \right) v_g \\
&\quad + \int_M (2s (d\dot{ϕ}, ds) - s^2 \Delta \dot{ϕ}) v_g \\
&= -4 \langle s, \mathbb{L}^c(\dot{ϕ}) \rangle = -4 \langle \mathbb{L}(s), \dot{ϕ} \rangle,
\end{align*}
where, we recall, \( L = \delta \delta D^* - d \) denotes the fourth order Lichnerowicz operator defined in Section 1.23, which here appears as the right hand side of (1.23.14). It follows that \( g \) is critical in \( M_{\Omega} \) iff \( L s_g = 0 \), iff \( s_g \) is a Killing potential of \( g \), meaning, we recall, that the hamiltonian vector field \( K := J \grad_s s_g = \grad_w s_g \) is a Killing vector field (equivalently, a (real) holomorphic vector field), cf. cf. Proposition 2.6.1 in Section 2.6.

3.3. Semi-positivity of the second derivative at critical points

Assume that \( g \) is an extremal Kähler metric on \((M, J)\), i.e. a critical point of the Calabi functional \( C \) in \( M_{\Omega} \). The second derivative of \( C \) at \( g \) is then given by the following proposition:

**Proposition 3.3.1.** The second derivative of \( C \) at a critical point \( g \) along the variations \( dd\dot{\phi}_1 \) and \( dd\dot{\phi}_2 \), where \( \dot{\phi}_1 \) and \( \dot{\phi}_2 \) are any two real functions on \( M \), is given by

\[
\dddot{C} = 8 \langle \delta \delta D^* - d \dot{\phi}_1, \delta \delta D^* - d \dot{\phi}_2 \rangle - 2 \langle L_K \dot{\phi}_1, L_K \dot{\phi}_2 \rangle,
\]

where \( K = J \grad_s \).

**Proof.** Without loss of generality, we can assume that \( \dot{\phi}_1 = \dot{\phi}_2 = \dot{\phi} \). Since \( g \) is extremal, we have \( D^* - ds = 0 \) at \( g \); it follows that the first derivative of \( \dddot{C} \) along the variation \( dd\dot{\phi} \) is given by

\[
\ddot{C} = -4 \langle D^* - d\dot{\phi}, (D^* - ds) \rangle,
\]

where \( (D^* - ds) \) denotes the first variation of \( D^* - ds \) along \( \omega = dd\dot{\phi} \) at the (extremal) metric \( g \). Since \( g \) is extremal, by Lemma 1.23.2 we have

\[
(D^* - ds) = -\frac{1}{2} \omega((\mathcal{L}_{\grad s} f)\cdot\cdot)
\]

\[
= D^*((\grad s)^\circ).
\]

It thus remains to compute \( (\grad s)^\circ = \grad s + \grad \hat{s} \), where \( \hat{s} \) is given by (3.2.6), whereas, for any (fixed) function \( f \), the first variation of \( \grad f \) is easily deduced from the identity: \( \grad f \cdot \omega = J df \), where the rhs is independent of \( g \). We thus get

\[
(\grad f)^\circ = (d^c f)^\circ \cdot dd\dot{\phi} \dot{\phi},
\]

so that

\[
(\grad s)^\circ = -d\Delta^2 \dot{\phi} - 2d((dd^c \dot{\phi}, \rho))
\]

\[
+ (d^c s)^\circ \cdot dd\dot{\phi} \dot{\phi};
\]

since \( K \) is Killing and preserves \( J \), the last term in the rhs can be written as follows:

\[
K_{\cdot dd^c \dot{\phi}} = \mathcal{L}_K d^c \dot{\phi} - d((d^c s, d^c \dot{\phi}))
\]

\[
= d^c \mathcal{L}_K \dot{\phi} - d((ds, d\dot{\phi}))
\]

\[
= d^c((d^c s, d\dot{\phi})) - d((ds, d\dot{\phi})) ;
\]
we thus finally get (by using (1.23.7) again):

\begin{equation}
(\text{grad } s)^\circ = -d \Delta_\phi \phi - 2d((d \phi, \rho)) - d((ds, d\phi)) + d^c((d^c s, d\phi))
\end{equation}

\begin{equation}
(3.3.5)
\end{equation}

\begin{equation}
=-2d(\delta \delta D^2 d\phi) + d^c((d^c s, d\phi)).
\end{equation}

Then, by (3.3.2) et (3.3.3):

\[
\bar{C} = 8 \langle D^- d\phi, D^- d\phi \rangle - 4 \langle D^- d\phi, D^- d^c((d^c s, d\phi)) \rangle
\]

\[
= 8 \langle L^*\phi, L\phi \rangle - 4 \langle D^- d\phi, D^- d^c((d^c s, d\phi)) \rangle
\]

\[
= 8 \langle L^*\phi, L\phi \rangle - 4 \langle \phi, \delta \delta D^2 d\phi \rangle.
\]

By (1.23.15), we have:

\[
-4 \langle \phi, \delta \delta D^2 d^c((d^c s, d\phi)) \rangle = 2\langle \phi, (d((d^c s, d\phi)), d\phi) \rangle
\]

\[
= 2 \langle d((d^c s, d\phi)), \phi d^c s \rangle
\]

\[
= 2 \langle (d^c s, d\phi), \delta (\phi d^c s) \rangle
\]

\[
= -2 \langle (d^c s, d\phi), (d^c s, d\phi) \rangle,
\]

where we have used the fact that \(d^c s\) is co-closed; since \((d^c s, d\phi) = \mathcal{L}_g\phi\), we eventually get (3.3.1). \(\square\)

The fact that \(\bar{C}\) is semi-positive at critical points is not directly visible from (3.3.1). Following Calabi in [45], it is convenient to consider the Calabi operators \(L^+\) and \(L^-\) introduced in Section 2.5, defined by (2.5.5)-(2.5.6).

Recall that the kernel of \(L^+\) is the space of complex functions \(F = f + ih\), \(f, h\) real, such that \(X = \text{grad } f + J\text{grad } h\) is a (real) holomorphic vector field, cf. Section 2.5.

In general, \(L^+\) and \(L^-\) are distinct. More precisely, it follows from (2.5.8)-(2.5.9) that \(L^+\) and \(L^-\) coincide if and only if \(\mathfrak{K} \equiv 0\), if and only if the scalar curvature \(s\) is constant. More generally,

**Proposition 3.3.2.** If the Kähler structure is extremal, then \(L^+\) and \(L^-\) commute.

**Proof.** If the Kähler structure is extremal, then \(\mathfrak{K}\) is Killing and preserves \(J\); we thus have the identity

\begin{equation}
(3.3.6)
\mathcal{L}_K \delta \delta D^2 dF = \delta \delta D^2 d\mathcal{L}_K F,
\end{equation}

for any \(F\). By (2.5.8) and (2.5.9), this directly implies that \(L^+\) and \(L^-\) commute. \(\square\)

We infer:

**Proposition 3.3.3 (E. Calabi [45]).** The second derivative \(\bar{C}\) at a critical metric \(g\) is semi-positive. The kernel of \(\bar{C}\) is the tangent space at \(g\) of the orbit of \(g\) under the action of \(\mathfrak{H}_{\text{red}}(M, J)\).

**Proof.** It follows from (2.5.8)-(2.5.9) and from (3.3.6) that (3.3.1) can be rewritten as

\begin{equation}
(3.3.7)
\bar{C} = 8 \langle L^+ \phi_1, L^- \phi_2 \rangle = 8 \langle L^- L^+ \phi_1, \phi_2 \rangle.
\end{equation}

The operators \(L^+\) and \(L^-\) are commuting, elliptic, self-adjoint, semi-positive operators on the compact manifold \(M\). This implies that the product
\[ L^+ L^+ = L^+ L^- \] is itself self-adjoint and semi-positive and that \( \ker(L^- L^+) = \ker L^- + \ker L^+ + \ker L^+. \) This means that a real function \( \phi \) belongs to \( \ker(L^- L^+) \) if and only if there exists a complex functions \( F \) in \( \ker L^+ \) such that \( \phi = F + \bar{F} = 2 \Re(F) \). By Proposition 2.5.1 this happens if and only if \( \phi \) is the real potential of a (real) holomorphic vector field \( X \) belonging to the ideal \( \mathfrak{h}_{\text{red}} \). Taking into account (3.4.3) which will be proved later, this is equivalent to \( \phi \) being the real potential of a (real) holomorphic vector field \( X \) in \( \mathfrak{h} \).

As observed by E. Calabi, Proposition 3.3.3 has the following direct corollary:

**Proposition 3.3.4.** The space of extremal Kähler metrics in \( \mathcal{M}_\Omega \) is a submanifold whose each connected component is an orbit of the reduced automorphism group \( H_{\text{red}}(M, J) \).

**Proof.** By Proposition 3.3.3, at any critical point of \( C \) the hessian of \( C \) is non-degenerate, even positive definite, in the directions transverse to the orbit of \( H_{\text{red}}(M, J) \). The space of extremal Kähler metrics is then a smooth submanifold of \( \mathcal{M}_\Omega \) and, at each point, its tangent space coincides with the tangent space of the orbit. Each orbit of \( H_{\text{red}}(M, J) \) is then open, hence coincides with the corresponding connected component. According to [60], the space of extremal Kähler metrics is actually either empty or connected. \( \square \)

### 3.4. Holomorphic vector fields on a compact extremal Kähler manifold

In this section, we assume that \( (M, g, J) \) is a connected, compact Kähler manifold and that the Kähler structure is extremal, i.e. that the hamiltonian vector field \( \mathbb{K} = J \text{grad}_g s \) is Killing.

The notation are those of Chapter 2. In particular we recall that:

1. the complex Lie algebra of infinitesimal automorphisms of \( (M, J) \), thought of as the Lie algebra of real vector fields preserving \( J \), has been denoted by \( \mathfrak{h} \) and is the Lie algebra of the (complex) Lie group \( H(M, J) \);
2. the Lie algebra of Killing vector fields, denoted by \( \mathfrak{k} \), is a (real) Lie subalgebra of \( \mathfrak{h} \) and is the Lie algebra of the (compact) group \( K(M, g) \);
3. the Lie algebra \( \mathfrak{a} \) of parallel (real) vector fields is a (complex) abelian complex Lie subalgebra of \( \mathfrak{h} \), contained in \( \mathfrak{k} \) and in the center of \( \mathfrak{h} \);
4. any element \( X \) of \( \mathfrak{h} \) can be uniquely written as

\[ X = X_H + \text{grad} f^X + J \text{grad} h^X, \]

where \( X_H \) is the dual of a harmonic 1-form and where \( f^X, h^X \) are real functions, normalized by \( \int_M f^X v_g = \int_M h^X v_g = 0 \), see Lemma 2.1.1 \( (f^X, \text{resp. } F^X = f^X + \mathbb{i} h^X, \text{ is called the real potential, resp. the complex potential, of } X) \);
(5) the ideal $h_{\text{red}}$ of element $X$ of $h$ such that $\alpha(X) = 0$ for all harmonic 1-forms $\alpha$ coincides with the space of elements of $h$ of the form $X = \text{grad} f + J\text{grad} h$, where $F = f + ih$ satisfies $L^+ F = 0$, see Proposition 2.5.1; $h_{\text{red}}$ is the Lie algebra of the reduced automorphism group $H_{\text{red}}(M, J)$;

(6) the space $k_{\text{ham}} = \mathfrak{k} \cap h_{\text{red}}$ of hamiltonian Killing vector fields coincides with the space of (real) vector fields of the form $J\text{grad} h$, where $h$ is a real function satisfying $L h = 0$, see proposition 2.6.1.

The following theorem is due to E. Calabi, cf. [45].

**Theorem 3.4.1.** For any compact extremal Kähler manifold, the complex Lie algebra $h$ admits the following orthogonal decomposition:

$$h = h^{(0)} \oplus (\oplus_{\lambda>0} h^{(\lambda)}),$$

where $h^{(0)}$ denotes the kernel of $L_K$, i.e. the centralizer of $K$ in $h$, and, for any $\lambda > 0$, $h^{(\lambda)}$ denotes the subspace of elements $X$ of $h$ such that $L_K X = \lambda JX$.

**Proof.** Let $X = X_H + \text{grad} f + J\text{grad} h$ be any element of $h$, where $X_H$ is the dual of the harmonic 1-form $\xi_H$. By (1.23.10), we have $\delta D^\perp \xi_H = JX_H \mathfrak{g} \rho$, so that, by using (1.23.16) and the fact that $J\xi_H$ is also harmonic, hence closed:

$$\delta D^\perp \xi_H = \delta(JX_H \mathfrak{g} \rho) = -\delta \rho, J\xi_H = \frac{1}{2}(d^c s, J\xi_H).$$

Now, $(d^c s, J\xi_H)$ is the inner product of a harmonic 1-form, $J\xi_H$, and the dual of a Killing vector field, $d^c s = K^\nu$; it is then constant, hence equal to zero since $K$ has zeros. It follows that $\delta \delta D^\perp \xi_H = 0$, whence also $\delta \delta D^\perp (df + d^c h) = 0$, as $\xi = \xi_H + df + Jdh$ is the dual of a (real) holomorphic vector field. By (2.5.7), the latter equality can be read: $\Re(L^+(F)) = 0$. Starting from $JX$ instead of $X$, we similarly get $\Im(L^+(F)) = 0$, hence, eventually, $L^+(F) = 0$. By Proposition 2.5.1, this implies that $\text{grad} f + J\text{grad} h$ and $X_H$ are separately (real) holomorphic. As already observed in Section 2.3, this implies that $X_H$ is parallel, hence belongs to $\mathfrak{a}$, whereas $\text{grad} f + J\text{grad} h$ certainly belongs to $h_0$. This completes the proof of (3.4.3), hence also of (3.4.4).
Since $L^-$ and $L^+$ commute, $L^-$ acts on $\ker L^+$; it readily follows from (2.5.8) and (2.5.9) that this action coincides with the action of $-iL_X$; since $L^-$ is self-adjoint and semi-positive, $\mathfrak{h}_{\text{red}}$ splits as

\begin{equation}
\mathfrak{h}_{\text{red}} = \mathfrak{h}_{\text{red}}^{(0)} \oplus (\oplus_{\lambda > 0} \mathfrak{h}^{(\lambda)}),
\end{equation}

where $\mathfrak{h}_{\text{red}}^{(0)}$ denotes the kernel of $L_X$ in $\mathfrak{h}_{\text{red}}$, whereas, for each $\lambda > 0$, $\mathfrak{h}^{(\lambda)}$ denotes the subspace of elements $X$ of $\mathfrak{h}$ such that $L_X X = \lambda X$ ($\mathfrak{h}^{(\lambda)} = \{0\}$ but for a finite, possibly empty, set of $\lambda$'s).

Since $L_X s = 0$ for any Killing vector field $X$, $\mathfrak{h}^{(0)}$ certainly contains the direct sum $\mathfrak{a} \oplus \mathfrak{t}_{\text{ham}} \oplus J\mathfrak{t}_{\text{ham}}$. In fact, both spaces coincide. Indeed, by (2.5.8) and (2.5.9) the restriction of $L_X$ to $\ker L^+$ coincides with the restriction of $\delta \delta D^{-1} d$ and we conclude by using Proposition 2.6.1. This completes the proof of (3.4.2).

\begin{proof}
By Theorem 3.4.1, when $(g, J)$ is extremal the kernel $\mathfrak{h}^{(0)}$ of $L_X$ in $\mathfrak{h}$ coincides with the sum $\mathfrak{t} + J\mathfrak{t} = \mathfrak{a} \oplus (\mathfrak{t}_{\text{ham}} \oplus J\mathfrak{t}_{\text{ham}})$ in $\mathfrak{h}$. On the other hand, $\mathfrak{t} = \mathfrak{a} \oplus \mathfrak{t}_0$ is the Lie algebra of the (connected) Lie group $\text{Iso}_0(M, g)$, which is a compact (real) subgroup of $H(M)$. The following theorem is the main result of [46] and the proof given here closely follows Calabi’s one in [46]:

**Theorem 3.5.1** (E. Calabi [46]). For any connected, compact extremal Kähler manifold, $K(M, g)$ is maximal among all compact, connected Lie subgroups of $H(M, J)$.

**Proof.** Let $\mathfrak{g}$ be the Lie algebra of a connected, compact Lie subgroup, $G$, of $H(M, J)$ containing $K(M, g)$. Suppose, for a contradiction, that there exists $X$ in $\mathfrak{g}$ that does not belong to $\mathfrak{t}$. Since $\mathfrak{t} = \mathfrak{a} \oplus \mathfrak{t}_{\text{ham}}$, we can assume that $X$ belongs to $J\mathfrak{t}_{\text{ham}} \oplus (\oplus_{\lambda > 0} \mathfrak{h}^{(\lambda)})$. Let $X = X_0 + \sum_{\lambda > 0} X_\lambda$ be the corresponding decomposition of $X$; then, $(L_X)^{2r} X = -\sum \lambda^{2r} X_\lambda$ for any positive integer $r$. We easily infer that each component of $X$ belongs to $\mathfrak{g}$, so that $\mathfrak{g} = \sum_{\lambda > 0} \mathfrak{g}^{(\lambda)}$, with $\mathfrak{g}^{(\lambda)} = \mathfrak{g} \cap \mathfrak{h}^{(\lambda)}$. We thus can assume that $X$ belongs to some $\mathfrak{g}^{(\lambda)}$, $\lambda > 0$, or to $J\mathfrak{t}_{\text{ham}} \subset \mathfrak{g}^{(0)}$.

Assume that $X$ belongs to some $\mathfrak{g}^{(\lambda)}$, $\lambda > 0$, and is not zero. Let $B$ denotes the Killing form of $\mathfrak{g}$. Since $\mathfrak{g}$ is the Lie algebra of a compact Lie
3.6. Kähler-Einstein metrics

A (compact) Kähler manifold whose scalar curvature is constant is certainly extremal since $\kappa = J\text{grad} s$ vanishes identically. Theorem 3.4.1 then implies:

**Theorem 3.6.1 (A. Lichnerowicz [135]).** For any compact Kähler manifold of constant scalar curvature, the complex Lie algebra of (real) holomorphic vector fields $\mathfrak{h}$ splits as

\[
\mathfrak{h} = \mathfrak{h}^{(0)} = \mathfrak{a} \oplus \mathfrak{t}_{\text{ham}} \oplus J\mathfrak{t}_{\text{ham}}.
\]

In general, the Killing form of a Lie algebra $\mathfrak{g}$ is the (symmetric) bilinear form on $\mathfrak{g}$ defined by $B^\mathfrak{g}(a, b) = \text{tr}(\text{ad}_a \circ \text{ad}_b)$, where $\text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g})$ stands for the usual ad-action of $\mathfrak{g}$ on itself given by the bracket, so that $\text{ad}_a b = [a, b]$, and the trace is over $\mathbb{R}$ or over $\mathbb{C}$, according as $\mathfrak{g}$ is a $\mathbb{R}$- or a $\mathbb{C}$-Lie algebra. If $\mathfrak{g}$ is the Lie algebra of a compact Lie group $G$, then $\mathfrak{g}$ can be given a $G$-invariant positive definite inner product, with respect to which all $\text{ad}_a$ are anti-symmetric. It follows that $B^\mathfrak{g}$ is semi-negative and that $a$ belongs to the kernel of $B^\mathfrak{g}$ if and only of $\text{ad}_a = 0$, i.e. if and only if $a$ belongs to the centre of $\mathfrak{g}$.
The decomposition 3.6.1 is a Lie algebra direct sum of the abelian Lie algebra \( \mathfrak{a} \) of parallel vector fields and of the complexification in \( \mathfrak{h} \) of the Lie algebra \( \mathfrak{t}_{\text{ham}} \) of hamiltonian Killing vector fields. In particular, \( \mathfrak{h} \) is a reductive complex Lie algebra.

**Proof.** (Cf. also Remark 2.6.1). Since \( \mathfrak{K} = J \text{grad}_g s_g = 0 \), for each \( \lambda > 0 \), \( \mathfrak{h}(\lambda) = \{0\} \) in (3.4.1) and (3.4.1) is then reduced to (3.6.1). We recall that a Lie algebra, say \( \mathfrak{l} \), is said to be reductive if its radical, i.e. its maximal solvable ideal, coincides with its center \( Z(\mathfrak{l}) \); then, either \( \mathfrak{l} \) is abelian, and we then have \( \mathfrak{l} = Z(\mathfrak{l}) \), or the derived ideal \( [\mathfrak{l}, \mathfrak{l}] \) of \( \mathfrak{l} \) is semi-simple; in both cases, \( \mathfrak{l} = Z(\mathfrak{l}) \oplus [\mathfrak{l}, \mathfrak{l}] \), cf. e.g. [105]. The Lie algebra of any compact Lie group is reductive: in this case, the Killing form is semi-negative, the center \( Z(\mathfrak{l}) \) coincides with the kernel of the Killing form and the Killing form of \( [\mathfrak{l}, \mathfrak{l}] \) is negative-definite. The complexification of the Lie algebra of a compact group is then reductive as well. By (3.6.1), in the case of a compact Kähler manifold of constant scalar curvature, \( \mathfrak{h} \) is a Lie algebra direct sum of \( \mathfrak{a} \), which is abelian, and of the complexification of \( \mathfrak{t}_{\text{ham}} \), which is the Lie algebra of a compact group; it is then reductive. \( \square \)

The set of compact Kähler manifold of constant scalar curvature includes the set of (compact) Kähler-Einstein manifolds, introduced in Section 1.19. We then have

**Theorem 3.6.2** (Y. Matsushima [148]). For any compact Kähler-Einstein manifold \( (M, g, J, \omega) \) of real dimension \( n = 2m \) and of (constant) scalar curvature \( s \), the complex Lie algebra of (real) holomorphic vector fields \( \mathfrak{h} \) is \( \{0\} \) if \( s < 0 \), is abelian and reduced to the space \( \mathfrak{a} \) of parallel vector fields if \( s = 0 \); if \( s > 0 \), \( \mathfrak{h} \) is of the form

\[
\mathfrak{h} = \mathfrak{t} \oplus J\mathfrak{t}.
\]

where \( \mathfrak{t} = \mathfrak{t}_{\text{ham}} \) denotes the space of Killing vector fields, with then coincides with the space of hamiltonian Killing vector fields. In particular, \( \mathfrak{h} \) is reductive.

Moreover, any positive eigenvalue \( \lambda \) of the riemannian laplacian \( \Delta \) acting on functions satisfies

\[
\lambda \geq \frac{s}{m},
\]

and \( \frac{s}{m} \) is an eigenvalue of \( \Delta \) if and only if \( \mathfrak{h} \neq \{0\} \). In this case, the eigenspace, \( \mathbf{E} \), of \( \frac{s}{m} \) is closed for the Poisson bracket and the map \( f \rightarrow Z^f := \text{grad}_\omega f = J\text{grad}_g f \) induces a Lie algebra isomorphism from \( \mathbf{E} \) to \( \mathfrak{t}_{\text{ham}} \).

**Proof.** If \( (g, J, \omega) \) is Kähler-Einstein with \( s < 0 \), then the Ricci tensor \( r \) is negative definite and then \( \mathfrak{h} = \{0\} \), cf. Remark 1.23.2; if \( s = 0 \), then \( r \equiv 0 \) and \( \mathfrak{h} = \mathfrak{a} \), cf Remark 1.23.2 again.

If \( s > 0 \), \( r \) is positive definite and it is a direct consequence of the Bochner formula (1.22.1) that a compact riemannian manifold with positive definite Ricci tensor has no non-zero harmonic 1-form (this fact holds for any compact riemannian manifold; in the Kähler case it has been refined by S. Kobayashi who proved that a compact Kähler manifold with positive
The definite Ricci tensor is simply-connected \([119]\); this refinement however is not needed here). In particular, \(M\) has no non-zero parallel 1-form, hence no non-zero parallel vector field: \((3.6.1)\) then reduces to \((3.6.2)\) and \(t = a \oplus \mathfrak{t}_{\text{ham}}\) reduces to \(\mathfrak{t}_{\text{ham}}\). In particular, \(\mathfrak{h}\) is reductive as the complexification of the Lie algebra of the compact Lie group \(K(M, g)\). Actually, \((3.6.2)\) can be obtained in a simpler way as a direct application of \((1.23.10)\) which, in the Kähler-Einstein case, simply reads

\[
\delta D^- \xi = \frac{1}{2} (\Delta \xi - \frac{s}{m} \xi).
\]

Since \(M\) has no non-zero harmonic 1-form, the riemannian dual \(\xi\) of any vector field \(X\) in \(\mathfrak{h}\) is of the form \(\xi = df + dc\), with \(\int_M f v_g = 0\), \(\int_M h v_g = 0\).

From \((3.6.4)\), we then get

\[
0 = \delta D^- \xi = \frac{1}{2} d(\Delta f - \frac{s}{m} f) + \frac{1}{2} d^c(\Delta h - \frac{s}{m} h).
\]

We infer that \(\text{grad}_g f\) and \(J \text{grad}_g h\) are separately in \(\mathfrak{h}\), in fact in \(J \mathfrak{t}_{\text{ham}}\) and in \(\mathfrak{t}_{\text{ham}}\) respectively. We also infer that \(J \text{grad}_g f\) is a non-zero element of \(\mathfrak{t}_{\text{ham}}\) if and only if \(f\) is a non-zero eigenfunction of \(\Delta\) for the eigenvalue \(\frac{s}{m}\). The map \(f \mapsto \text{grad}_\omega f = J \text{grad}_g f\) then induces a linear isomorphism from the space, \(E\), of (real) functions \(f\) satisfying \(\Delta f = \frac{s}{m} f\) onto the space \(\mathfrak{t}_{\text{ham}}\) of hamiltonian Killing vector fields. Moreover, because of \((1.2.11)\) we infer that \(E\) is closed with respect to the Poisson bracket and that the isomorphism \(E \cong \mathfrak{t}\) is therefore a Lie algebra isomorphism from \(E\), equipped with the Poisson bracket, to the Lie algebra \(\mathfrak{t}\) of Killing vector fields. Finally, by integrating \((3.6.4)\) over \(M\) when \(\xi = df\) and \(f\) is an eigenfunction of \(\Delta\) with respect to any positive eigenvalue \(\lambda\), we get

\[
\int_M |D^- df|^2 v_g = \frac{1}{2} \lambda (\lambda - \frac{s}{m}) \int_M |f|^2 v_g.
\]

We infer that any positive eigenvalue \(\lambda\) of \(\Delta\) satisfies \((3.6.3)\) and that \(\frac{s}{m}\) is an eigenvalue of \(\Delta\) if and only if \(\mathfrak{t}_{\text{ham}} \neq \{0\}\). \(\square\)

Theorems 3.6.2 and 3.6.1 provide an obstruction to the existence of Kähler metrics with constant scalar curvature — a fortiori for Kähler-Einstein metrics — which can be expressed as follows:

A compact complex manifold whose automorphism group \(\text{Aut}(M, J)\) is not reductive admits no Kähler metric with constant scalar curvature.

Examples of such complex manifolds are provided in Section 6.6 and in Chapter 10.

Remark 3.6.1. Part of the argument in the proof of Theorem 3.6.2 is a special case of the following general result:

Proposition 3.6.1 (A. Lichnerowicz [135]). Let \((M, g, J, \omega)\) be a compact Kähler manifold, whose Ricci tensor \(r\) satisfies

\[
r \geq kg,
\]

for some positive real number \(k\). Denote by \(\lambda_1\) the smallest positive eigenvalue of the riemannian Laplace operator \(\Delta\) and by \(E_1\) the corresponding eigenspace. Then,

\[
\lambda_1 \geq 2k.
\]
Moreover, if \( \lambda_1 = 2k \) then all elements of \( E_1 \) are Killing potentials.

**Proof.** Let \( f \) be any non-zero element of \( E_1 \). From (1.23.10) applied to \( \xi = df \) we infer \( \delta D^- df = \frac{\lambda_1}{2} df - r(\text{grad}_g f) \). By considering the global inner product of both sides with \( df \) and by using (3.6.7), we get

\[
\int_M |D^- df|^2 v_g \leq \frac{1}{2} (\lambda_1 - 2k) \int_M |df|^2 v_g.
\]

Since \( f \) is not constant, we readily infer (3.6.8). Moreover, if equality holds, then \( D^- df = 0 \), meaning that \( J\text{grad}_g f \) is (real) holomorphic, hence Killing. \( \square \)
CHAPTER 4

More on the geometry of the space of Kähler metrics

4.1. The space of relative Kähler potentials

The space \( \mathcal{M}_\Omega \) of Kähler metrics within a fixed Kähler class \( \Omega \), as defined in Section 3, is closely related to the space of relative Kähler potentials, \( \tilde{\mathcal{M}}_{\omega_0} \), defined as the space of real functions \( \phi \) on \( M \) such that \( \omega_\phi := \omega_0 + dd^c \phi \) is positive definite, for some chosen element \( \omega_0 \) in \( \mathcal{M}_\Omega \) (the corresponding Riemannian metric will be denoted by \( g_0 \)).

Notice that if \( \phi \) belongs to \( \tilde{\mathcal{M}}_{\omega_0} \), so does \( t\phi \) for each real number \( t \) in the interval \([0, 1]\), as \( \omega_{t\phi} = \omega_0 + tdd^c \phi = (1-t)\omega_0 + t\omega_\phi \).

As a (contractible) open subset of \( C^\infty(M, \mathbb{R}) \), \( \tilde{\mathcal{M}}_{\omega_0} \) is a Fréchet manifold modeled on \( C^\infty(M, \mathbb{R}) \) and, for each \( \phi \) in \( \tilde{\mathcal{M}}_{\omega_0} \), the tangent space \( T_\phi \tilde{\mathcal{M}}_{\omega_0} \) is naturally identified with \( C^\infty(M, \mathbb{R}) \). In particular, any \( f \) in \( C^\infty(M, \mathbb{R}) \) can be regarded as a constant vector field on \( \tilde{\mathcal{M}}_{\omega_0} \). The space of constant vector fields on \( \tilde{\mathcal{M}}_{\omega_0} \) generates the tangent bundle \( T\tilde{\mathcal{M}}_{\omega_0} \) in the sense that for any \( \phi \) in \( \tilde{\mathcal{M}}_{\omega_0} \) any element of \( T_\phi \tilde{\mathcal{M}}_{\omega_0} \) is the value at \( \phi \) of a (uniquely) defined constant vector field. Moreover for any two elements \( f_1, f_2 \) of \( C^\infty(M, \mathbb{R}) \), the corresponding constant vector fields, still denoted by \( f_1, f_2 \), commute.

We observe that the definition of \( \tilde{\mathcal{M}}_{\omega_0} \) depends upon the choice of a basepoint \( \omega_0 \) of \( \mathcal{M}_\Omega \). In particular, the group \( H(M, J) \) does not act directly on \( \tilde{\mathcal{M}}_{\omega_0} \), since \( \omega_0 \) is not preserved in general by the action of \( H(M, J) \) on \( \mathcal{M}_\Omega \).

The additive group \( \mathbb{R} \) acts freely on \( \tilde{\mathcal{M}}_{\omega_0} \), by \( a \cdot \phi = \phi + a \), for any \( a \) in \( \mathbb{R} \) and \( \mathcal{M}_\Omega \) is then naturally identified with the quotient \( \tilde{\mathcal{M}}_{\omega_0}/\mathbb{R} \). The natural projection of \( \tilde{\mathcal{M}}_{\omega_0} \) onto \( \mathcal{M}_\Omega \) will be denoted by \( \pi \).

It may be convenient to realize \( \mathcal{M}_\Omega \) as a submanifold of \( \tilde{\mathcal{M}}_{\omega_0} \), so that \( \tilde{\mathcal{M}}_{\omega_0} \) is identified with the product \( \mathcal{M}_\Omega \times \mathbb{R} \). This can be done as follows [74].

We first observe that the (smooth) assignment \( \phi \mapsto \nu_\phi \) of a \( n \)-form \( \nu_\phi \) on \( M \) to each element \( \phi \) of \( \tilde{\mathcal{M}}_{\omega_0} \) can be viewed as a 1-form, \( \tilde{\nu} \) say, on \( \tilde{\mathcal{M}}_{\omega_0} \), via the pairing

\[
\tilde{\nu}_\phi(f) = \int_M f \nu_\phi,
\]

for each \( f \) in \( C^\infty(M, \mathbb{R}) = T_\phi \tilde{\mathcal{M}}_{\omega_0} \). In particular, let \( \tau \) be the 1-form on \( \tilde{\mathcal{M}}_{\omega_0} \) defined by

\[
\tau_\phi(f) = \frac{1}{V_\Omega} \int_M f \nu_\phi,
\]

where \( V_\Omega \) is the volume of \( \Omega \).
for each φ in $\tilde{M}_{\omega_0}$ and each $f$ in $T_\phi \tilde{M}_{\omega_0} = C^\infty(M, \mathbb{R})$, where $v_\phi$ stands for the volume form of $\omega_\phi$.

It is easily checked that $\tau$ is closed. Indeed, choose any two elements $f_1, f_2$ of $C^\infty(M, \mathbb{R})$, viewed as (commuting, constant) vector fields on $\tilde{M}_\Omega$ (see above). By using (3.2.4), we have that

$$d\tau(f_1, f_2) = f_1 \cdot \tau(f_2) - f_2 \cdot \tau(f_1) = \frac{1}{V_\Omega} \int_M (f_1 \Delta f_2 - f_2 \Delta f_1) v_\phi = 0.$$ 

Since $\tilde{M}_{\omega_0}$ is contractible, $\tau$ admits a unique primitive, denoted by $I_{\omega_0}$, equal to 0 at $\phi = 0$. We thus have

$$\tag{4.1.3} (d I_{\omega_0})_\phi(f) = \frac{1}{V_\Omega} \int_M f v_\phi,$$

for any $\phi$ in $\tilde{M}_{\omega_0}$ and any $f$ in $T_\phi \tilde{M}_{\omega_0} = C^\infty(M, \mathbb{R})$, and

$$\tag{4.1.4} I_{\omega_0}(\phi) = \int_0^1 \tau_{\phi_t}(\phi_t) dt,$$

where $t \in [0, 1] \rightarrow \phi_t$ is any curve connecting $0 = \phi_0$ to $\phi = \phi_1$ in $\tilde{M}_{\omega_0}$. By choosing the curve $t \mapsto t\phi$, we have $\dot{\phi}_t = \phi$ for all $t$, so that:

$$\tag{4.1.5} I_{\omega_0}(\phi) = \frac{1}{V_\Omega} \int_M \phi \left( \int_0^1 v_t dt \right),$$

where $v_t$ stands for the volume form of $\omega_t := \omega_0 + tdd^c \phi$, i.e. $v_t = \frac{1}{m!} (\omega_0 + tdd^c \phi)^m$.

**Lemma 4.1.1.** The function $I_{\omega_0}$ defined by (4.1.5) has the following alternative expression

$$\tag{4.1.6} I_{\omega_0}(\phi) = \frac{1}{V_\Omega} \int_M \phi \sum_{j=0}^m \omega_\phi^j \wedge \omega_0^{m-j} \over (m+1)!$$

for any $\phi$ in $\tilde{M}_\Omega$, with $\omega_\phi = \omega_0 + dd^c \phi$.

**Proof.** The rhs of (4.1.6) is evidently equal to 0 when $\phi \equiv 0$. It is then sufficient to check that it is a primitive of $\tau$. By differentiating the rhs of (4.1.6), we get

$$
\begin{align*}
\frac{1}{V_\Omega} \int_M f \sum_{j=0}^m \omega_\phi^j \wedge \omega_0^{m-j} &+ \phi \sum_{j=1}^m j \omega_\phi^{j-1} \wedge \omega_0^{m-j} \wedge dd^c f \\
= \frac{1}{V_\Omega} \int_M f \sum_{j=0}^m \omega_\phi^j \wedge \omega_0^{m-j} &+ \sum_{j=1}^m j \omega_\phi^{j-1} \wedge \omega_0^{m-j} \wedge dd^c \phi \\
= \frac{1}{V_\Omega} \int_M f \sum_{j=0}^m \omega_\phi^j \wedge \omega_0^{m-j} &+ j (\omega_\phi^{j-1} \wedge \omega_0^{m-j}) \wedge (\omega_\phi - \omega_0) \\
= \frac{1}{V_\Omega} \int_M f \frac{\omega_\phi^m}{m!} = \tau_\phi(f).
\end{align*}
$$

The functional $I_{\omega_0}$ defined by (4.1.5)-(4.1.6) is $\mathbb{R}$-equivariant:

$$\tag{4.1.7} I_{\omega_0}(\phi + a) = I_{\omega_0}(\phi) + a,$$
for any \( \phi \) in \( \tilde{M}_{\omega_0} \) and any \( a \) in \( \mathbb{R} \), as \( \frac{1}{V_{\Omega}} \int_M v_t = 1 \) for all \( t \). In particular, \( \mathbb{I}_{\omega_0} \) is everywhere regular and thus makes \( \tilde{M}_{\omega_0} \) into a fibration over \( \mathbb{R} \) whose fibers — the level sets of \( \mathbb{I}_{\omega_0} \) — are all isomorphic to \( \mathcal{M}_\Omega \) and can be viewed as the graphs of parallel sections of \( \pi : \tilde{M}_{\omega_0} \to \mathcal{M}_\Omega \) with respect to the connection determined by \( \tau \).

In view of the above, the space \( \mathcal{M}_\Omega \) can be naturally regarded as a subspace of \( \mathcal{M}_\Omega \) via the identification

\[
\mathcal{M}_\Omega = \mathbb{I}_{\omega_0}^{-1}(0).
\]

Within this identification, any \( \omega \) in \( \mathcal{M}_\Omega \) is unambiguously written as \( \omega_0 + dd^c\phi \), where the relative Kähler potential \( \phi \) is normalized by \( \mathbb{I}_{\omega_0}(\phi) = 0 \), cf. [74]. For any curve \( \omega_t = \omega_0 + dd^c\phi_t \) in \( \mathcal{M}_\Omega \), with \( \mathbb{I}_{\omega_0}(\phi_t) = 0 \), the derivative \( \phi_t \) will then automatically satisfy Condition (3.1.3).

The functional \( \mathbb{I}_{\omega_0} \) has been first introduced in Bando-Mabuchi’s paper [23]. In view of (4.1.8) and of its role in Donaldson’s paper [74], it will be referred to as the Donaldson gauge functional relative to \( \omega_0 \).

Notice that the map \( \phi \mapsto (\omega_0 = \omega_0 + dd^c\phi, \mathbb{I}_{\omega_0}(\phi)) \) is a diffeomorphism of Fréchet manifolds from \( \tilde{M}_{\omega_0} \) to the product \( \mathcal{M}_\Omega \times \mathbb{R} \), whose inverse is the map \( (\omega = \omega_0 + dd^c\phi, a) \mapsto \phi - \mathbb{I}_{\omega_0}(\phi) + a \).

**Remark 4.1.1.** If another base-point \( \omega_0' = \omega_0 + dd^c\psi \) is chosen in \( \mathcal{M}_\Omega \), the space \( \tilde{M}_{\omega_0'} \) of relative Kähler potentials with respect to \( \omega_0' \) is deduced from \( \tilde{M}_{\omega_0} \) by the translation \( \phi \mapsto \phi' = -\psi + \phi \) in \( C^\infty(M, \mathbb{R}) \). We thus have

\[
\mathbb{I}_{\omega_0'}(\phi') = \mathbb{I}_{\omega_0}(\phi' + \psi) - \mathbb{I}_{\omega_0}(\psi),
\]

for any \( \phi' \) in \( \tilde{M}_{\omega_0'} \). Notice that \( \mathbb{I}_{\omega_0}(\psi) = -\mathbb{I}_{\omega_0'}(-\psi) \).

### 4.2. Aubin functionals

The functionals \( I_{\omega_0} \) and \( J_{\omega_0} \) on \( \mathcal{M}_\Omega \) considered in this section were introduced by T. Aubin in [20] and have been extensively used since then in the literature. These depend on the choice of a base-point \( \omega_0 \) and are then defined by

\[
I_{\omega_0}(\omega_0) = \frac{1}{V_{\Omega}}(\int_M \phi v_{g_0} - \int_M \phi v_{g_\phi}),
\]

\[
J_{\omega_0}(\omega_0) = \int_0^1 \frac{I_{\omega_0}(\omega_{t\phi})}{t} dt
\]

for any \( \omega_0 = \omega_0 + dd^c\phi \) in \( \mathcal{M}_\Omega \), where \( \omega_{t\phi} = \omega_0 +tdt^c\phi \), \( 0 \leq t \leq 1 \), is the linear interpolation between \( \omega_0 \) and \( \omega_\phi \). Here, as usual, \( g_\phi, g_{t\phi} \) denote the corresponding riemannian metrics and \( v_{g_\phi}, v_{g_{t\phi}} \) the corresponding volume forms.

**Proposition 4.2.1 ([17], [23], [186], [187]).** The Aubin energy functionals \( I_{\omega_0} \) and \( J_{\omega_0} \) admit the following alternative expressions:

\[
I_{\omega_0}(\omega_\phi) = \frac{1}{V_{\Omega}} \int_M d\phi \wedge dd^c\phi \wedge \sum_{i=0}^{m-1} \omega_0^{m-1-i} \wedge \omega_\phi^i / m!,
\]

\[
I_{\omega_0}(\omega_\phi) = \frac{1}{V_{\Omega}} \int_M d\phi \wedge dd^c\phi \wedge \sum_{i=0}^{m-1} \omega_0^{m-1-i} \wedge \omega_\phi^i / m!,
\]
and

\[(4.2.4) \quad J_{\omega_0}(\omega_\phi) = \frac{1}{V_\Omega} \int_M d\phi \wedge d^c \phi \wedge \sum_{i=0}^{m-1} \frac{(m-i) \omega_0^{m-i-1} \wedge \omega^i}{(m+1)!},\]

for any \(\omega_\phi = \omega_0 + dd^c \phi\) in \(M_\Omega\). In particular, \(I_{\omega_0}\) and \(J_{\omega_0}\) satisfy

\[(4.2.5) \quad 0 \leq \frac{1}{(m+1)} I_{\omega_0}(\omega) \leq \frac{m}{(m+1)} J_{\omega_0}(\omega),\]

for any \(\omega\) in \(M_\Omega\), with equality — in any of the above three inequalities — if and only if \(\omega = \omega_0\).

**Proof.** From (4.2.1) we get

\[
I_{\omega_0}(\omega_\phi) = \frac{1}{V_\Omega} \int_M \phi(\omega_0 - \omega_\phi) \wedge \sum_{i=0}^{m-1} \frac{\omega_0^{m-i-1} \wedge \omega^i}{m!}
\]

\[
= -\frac{1}{V_\Omega} \int_M \phi dd^c \phi \wedge \sum_{i=0}^{m-1} \frac{\omega_0^{m-i} \wedge \omega^i}{m!}
\]

\[
= \frac{1}{V_\Omega} \int_M d\phi \wedge d^c \phi \wedge \sum_{i=0}^{m-1} \frac{\omega_0^{m-1-i} \wedge \omega^i}{m!},
\]

which is (4.2.3). Similarly, from (4.2.7) and (4.1.6), we get

\[
J_{\omega_0}(\omega_\phi) = \frac{1}{V_\Omega} \int_M \phi \left( \frac{\omega^m_0}{m!} - \sum_{i=0}^{m-1} \frac{\omega^{m-i}_0 \wedge \omega^i}{(m+1)!} \right)
\]

\[
= \frac{1}{V_\Omega} \int_M \phi \sum_{i=0}^{m-1} \frac{\omega^{m-1-i}_0 \wedge \omega^i_0 - \omega^{m-i}_0}{(m+1)!}
\]

\[
= -\frac{1}{V_\Omega} \int_M \phi dd^c \phi \wedge \sum_{i=0}^{m-1} \frac{\omega^{m-1-i}_0 \wedge \omega^i_0 + \omega^{m-2-i}_0 \wedge \omega^i_0 + \ldots + \omega^{m-1-i}_0}{(m+1)!}
\]

\[
= \frac{1}{V_\Omega} \int_M d\phi \wedge d^c \phi \wedge \sum_{i=0}^{m-1} \frac{(m-i)\omega^{m-1-i}_0 \wedge \omega^i_0}{(m+1)!},
\]

which is (4.2.4). For the rest of the proof of Proposition 4.2.1 we need the following lemma:

**Lemma 4.2.1.** For any \(\phi\) in \(\tilde{M}_{\omega_0}\) and for each \(i = 0, \ldots, m-1\), we have

\[
(4.2.6) \quad \int_M d\phi \wedge d^c \phi \wedge \frac{\omega^{m-1-i}_0 \wedge \omega^i_0}{(m-1)!} = \int_M |d\phi|_{g^{(i)}} v_{g^{(i)}},
\]

where \(g^{(i)}\) denotes the unique hermitian metric on \((M, J)\) — in general non-Kähler for \(0 < i < m-1\) — whose Kähler form, \(\omega^{(i)}\), satisfies \((\omega^{(i)})^{m-1} = \omega^{m-1-i}_0 \wedge \omega^i_0\).

**Proof.** The existence of \(\omega^{(i)}\) is easily established by considering, for any \(x\) in \(M\), a \(J\)-adapted basis \(\{e_1, Je_1, \ldots, e_m, Je_m\}\) of \(T_x M\), with respect to which \(\omega_0(x) = \sum_{\tau=1}^{m} e^*_\tau \wedge Je^*_\tau\) and \(\omega_\phi(x) = \sum_{\tau=1}^{m} \lambda_\tau e^*_\tau \wedge Je^*_\tau\) (where \(\{e_1, Je_1, \ldots, e_m, Je_m\}\) denotes the dual basis of \(T^*_x M\) and the \(\lambda_\tau\)'s are all positive real numbers). We then get \(\omega^{(i)}(x) = \sum_{\tau=1}^{m} \mu^{(i)}_\tau e^*_\tau \wedge Je^*_\tau\), with
\[ \mu_r^{(i)} = \frac{c_i^{1/m-1}}{\sigma_i(\lambda_1, \ldots, \lambda_r, \ldots, \lambda_m)}, \]

where \( \sigma_i(\lambda_1, \ldots, \lambda_r, \ldots, \lambda_m) \) stands for the \( i \)-th elementary symmetric function of the \((m-1)\) numbers obtained by deleting \( \lambda_r \) from \( \lambda_1, \ldots, \lambda_m \), and \( c_i = \binom{m-1}{i} \prod_{s=1}^{m} \sigma_s(\lambda_1, \ldots, \hat{\lambda_s}, \ldots, \lambda_m) \). Observe that \( \omega^{(0)} = \omega_0 \) and \( \omega^{(m-1)} = \omega_{\phi} \), while \( \omega^{(i)} \) is not closed in general for \( i = 1, \ldots, m-2 \). Then, (4.2.6) readily follows from \( (\omega^{(i)})^{m-1} = \omega_0^{m-1} \wedge \omega_{\phi}^i \).

The uniqueness of \( \omega^{(i)} \) follows from the general fact that \( \omega_1^{m-1} = \omega_2^{m-1} \) implies \( \omega_1 = \omega_2 \) for any two positive \( J \)-invariant 2-forms, \( \omega_1, \omega_2 \) (easy verification by diagonalizing \( \omega_2 \) with respect to \( \omega_1 \) as above).

The second part of Proposition 4.2.1 readily follows from (4.2.3)-(4.2.4) upon using (4.2.6), which ensures the positivity of all terms in the rhs of (4.2.3)-(4.2.4) for any non-constant \( \phi \) in \( \mathcal{M}_{\omega_0} \).

The Aubin functionals \( I_{\omega_0} \) and \( J_{\omega_0} \) are related to the Donaldson gauge functional \( \mathbb{I}_{\omega_0} \) defined in Section 4.1 via the next statement:

**Proposition 4.2.2.** For any \( \phi \) in \( \mathcal{M}_{\omega_0} \), we have

\[
\frac{1}{V_{\Omega}} \int_M \phi v_{g_0} = J_{\omega_0}(\omega_\phi) + \mathbb{I}_{\omega_0}(\phi),
\]

(4.2.7)

\[
- \frac{1}{V_{\Omega}} \int_M \phi v_{g_\phi} = I_{\omega_0}(\omega_\phi) - J_{\omega_0}(\omega_\phi) - \mathbb{I}_{\omega_0}(\phi).
\]

In particular, if \( \phi \) is normalized by

\[ \mathbb{I}_{\omega_0}(\phi) = 0, \]

(4.2.8)

\( \int_M \phi v_{g_0} \) and \( -\int_M \phi v_{g_\phi} \) are both positive, unless \( \phi \equiv 0 \), and we have

\[ \frac{1}{m} \int_M \phi v_{g_0} \leq -\int_M \phi v_{g_\phi} \leq m \int_M \phi v_{g_0}, \]

with equality in both sides if and only if \( \phi \equiv 0 \).

**Proof.** The first equality in (4.2.7) is a direct consequence of (4.1.5) and (4.2.2); the second equality then follows from (4.2.1). If \( \phi \) is normalized by (4.2.8), we infer

\[ \frac{1}{V_{\Omega}} \int_M \phi v_{g_0} = J_{\omega_0}(\omega_\phi), \]

(4.2.10)

\[ - \frac{1}{V_{\Omega}} \int_M \phi v_{g_\phi} = (I - J)_{\omega_0}(\omega_\phi). \]

Then, (4.2.9) easily follows from (4.2.5). By Proposition 4.2.1, in (4.2.9), equality holds in either inequalities if and only if \( \phi \) is constant, hence identically zero because of (4.2.8).

**Remark 4.2.1.** When \( \mathcal{M}_{\Omega} \) is identified to \( \mathbb{I}_{\omega_0}^{-1}(0) \), cf. Section 4.1, by Proposition 4.2.2, both functionals \( \phi \mapsto \frac{1}{V_{\Omega}} \int_M \phi v_{g_0} \) and \( \phi \mapsto -\frac{1}{V_{\Omega}} \int_M \phi v_{g_\phi} \) on \( \mathcal{M}_{\omega_0} \) can be viewed as distance functions from \( \omega_0 \) on \( \mathcal{M}_{\Omega} \), as well as \( I_{\omega_0}, J_{\omega_0} \) and \( (I - J)_{\omega_0} := I_{\omega_0} - J_{\omega_0} \). By Proposition 4.2.1 and Proposition 4.2.2, all these distance functions are pairwise uniformly bounded by constants which only depend on \( m \).
Proposition 4.2.3. For any $\omega_\phi = \omega_0 + dd^c \phi$ in $\mathcal{M}_\Omega$, the derivatives of $I_{\omega_0}$, $J_{\omega_0}$ and $(I-J)_{\omega_0}$ at $\omega_\phi$ are given by

$$dI_{\omega_0}(f) = \frac{1}{V_\Omega} \left( \int_M f v_{g_0} - \int_M f v_{g_\phi} + \int_M f \Delta_\phi v_{g_\phi} \right),$$

$$dJ_{\omega_0}(f) = \frac{1}{V_\Omega} \left( \int_M f v_{g_0} - \int_M f v_{g_\phi} \right),$$

$$d(I-J)_{\omega_0}(f) = \frac{1}{V_\Omega} \int_M f \Delta_\phi v_{g_\phi},$$

for any $\omega_\phi = \omega_0 + dd^c \phi$ in $\mathcal{M}_\Omega$ and any $f$ in $T_{\omega_\phi} \mathcal{M}_\Omega = C^\infty(M, \mathbb{R})/\mathbb{R}$, where $\Delta_\phi$ denotes the riemannian laplacian relative to $g_\phi$.

Proof. The first identity follows from (4.2.1) and (3.2.4); the second identity follows from (4.2.7) and (4.1.3); the third identity follows from the previous two ones. □

4.3. The Mabuchi metric

The space $\tilde{\mathcal{M}}_{\omega_0}$ comes equipped with a natural (weak) riemannian metric, whose inner product at $\phi$ is defined by

$$\langle f_1, f_2 \rangle_\phi = \int_M f_1 f_2 v_{g_\phi},$$

for any pair $f_1, f_2$ of real functions on $M$, viewed as elements of $T_{\phi} \tilde{\mathcal{M}}_{\omega_0}$, where $g_\phi = \pi(\phi)$ denotes the metric determined by $\phi$, i.e. the metric whose Kähler form is $\omega = \omega_0 + dd^c \phi$. The space $\mathcal{M}_\Omega$, identified to the level set $I_{\omega_0}^{-1}(0)$ in $\tilde{\mathcal{M}}_{\omega_0}$, cf. Section 4.1, is equipped with the induced riemannian metric: at any $g$ in $\mathcal{M}_\Omega$, the corresponding inner product is then given by (4.3.1) again, for any $f_1, f_2$ in $T_g \mathcal{M}_\Omega$, hence for any (smooth) real functions $f_1, f_2$ on $M$ satisfying (3.1.3).

The above riemannian metrics on $\tilde{\mathcal{M}}_{\omega_0}$ and $\mathcal{M}_\Omega$ has been first introduced and worked out by T. Mabuchi in [142]. They will therefore be referred to as the Mabuchi metrics.

Notice that $\tilde{\mathcal{M}}_{\omega_0}$, equipped with its Mabuchi metric, is a riemannian product of $\mathcal{M}_\Omega$, equipped with its Mabuchi metric, with $\mathbb{R}$, equipped with its standard metric, whereas $\pi : \mathcal{M}_\Omega \to \mathcal{M}_\Omega$ is a riemannian submersion.

The Mabuchi metric and the Poisson bracket defined in Section 1.2 are linked together by:

$$\langle \{f_3, f_1\}, f_2 \rangle_\phi + \langle f_1, \{f_3, f_2\} \rangle_\phi = 0,$$

for each $\phi$ in $\tilde{\mathcal{M}}_{\omega_0}$ and for any $f_1, f_2, f_3$ in $C^\infty(M, \mathbb{R})$, viewed as elements of $T_{\phi} \tilde{\mathcal{M}}_{\omega_0}$, where the Poisson bracket is relative to the symplectic form $\omega = \omega_0 + dd^c \phi$. This identity readily follows from the following expressions of the Poisson bracket in the Kähler context:

$$\{f_1, f_2\} = \Lambda(df_1 \wedge df_2) = \delta(f_1 d^c f_2) = -d^c(f_1 df_2),$$

These in turn are easily checked by using the Kähler identities (1.14.1).
4.4. The Levi-Civita connection of the Mabuchi metric

In an infinite dimensional context, there is a priori no guarantee that a (weak) riemannian metric should admit a Levi-Civita connection, as the usual Koszul formula only provides an element of the dual of the tangent space, which in general cannot be identified with an element of the tangent space itself. In the present situation however, a Levi-Civita connection for the Mabuchi metric exists on $\tilde{M}_{\omega_0}$, then on $\mathcal{M}_\Omega = T_{\omega_0}(0)$, denoted by $\mathcal{D}$ in both cases, according to the following proposition.

**Proposition 4.4.1.** The Mabuchi metric on $\tilde{M}_{\omega_0}$ admits a unique Levi-Civita connection, $\mathcal{D}$, defined at $\phi$ by

\begin{equation}
\mathcal{D}_f\xi = \xi_\phi(f) - (d\xi, df)_g,
\end{equation}

for any vector field $\xi$ on $\tilde{M}_{\omega_0}$ and any $f$ in $T_{\phi}\tilde{M}_{\omega_0}$, where $g = \pi(\phi)$ denotes the associated metric in $\mathcal{M}_\Omega$, and $\xi_\phi(f) = \frac{d}{dt}|_{t=0}^{\xi_0 + tf}$ denotes the variation of $\xi$ along $f$ at $\phi$.

**Proof.** We first show that a linear connection on $\tilde{M}_{\omega_0}$ which is torsion-free and preserves the Mabuchi metric.

For any vector field $\xi$, for any vector field $f$, $\xi$ is a constant vector field, that is to say $\xi(\phi) = f_2$ for any $\phi$ in $\mathcal{M}_{\omega_0}$, where $f_2$ is a fixed fonction on $M$. Constant vector fields on $\tilde{M}_{\omega_0}$ pairwise commute and the Koszul formula, applied to the constant vector fields $f_1, f_2, f_3$, then reduces to

$$2\langle \mathcal{D}_f f_1, f_3 \rangle = f_1 \cdot \langle f_2, f_3 \rangle + f_2 \cdot \langle f_1, f_3 \rangle - f_3 \cdot \langle f_1, f_2 \rangle.$$

By using (3.2.4), we get

$$2\langle \mathcal{D}_f f_1, f_3 \rangle = -\langle f_2 f_3, \Delta f_1 \rangle - \langle f_1 f_3, \Delta f_2 \rangle + \langle f_1 f_2, \Delta f_3 \rangle$$

$$= \langle \Delta(f_1 f_2) - f_1 \Delta f_2 - f_2 \Delta f_1, f_3 \rangle$$

$$= -2(df_1, df_2, f_3).$$

In the case when $\xi$ is a constant vector field, $\mathcal{D}_f \xi$ is then necessarily of the form

\begin{equation}
\mathcal{D}_f \xi = -(d\xi, df)_g.
\end{equation}

For any vector field $\xi$, and any constant vector field $f$, $\xi$ on $\tilde{M}_{\omega_0}$, we then have (since $\mathcal{D}$ should preserve the Mabuchi metric):

$$\langle \mathcal{D}_f \xi, f_3 \rangle = f \cdot \langle \xi, f_3 \rangle - \langle \xi, \mathcal{D}_f f_3 \rangle$$

$$= f \cdot \int_M \xi(\phi) f_3 v_g + \int_M \xi(\phi) f(df, df_3)_g v_g$$

$$= \int_M \hat{\xi}_0(f_3 v_g - \int_M \xi(\phi) f_3 \Delta_g(f) v_g + \int_M \delta_g(\xi(\phi) df)f_3 v_g$$

$$= \int_M \hat{\xi}_0(f_3 - (d\xi(\phi), df)_g) v_g.$$  

This shows that $\mathcal{D}$ is necessarily given by (4.4.1). Conversely, it is easy to check that the rhs of (4.4.1) defines a linear connection on $\tilde{M}_{\omega_0}$, which is torsion-free and preserves the Mabuchi metric. \qed
The induced connection on $\mathcal{M}_\Omega$, still denoted by $\mathcal{D}$, is also given by (4.4.1) for any vector field $\xi$ on $\mathcal{M}_\Omega$. Notice that, $\xi$ is then an assignment $g \mapsto \xi(g)$, which associates $\xi(g)$ in $C^0_0(M, \mathbb{R})$ to any $g$ in $\mathcal{M}_\Omega$ and that, by (3.2.4), the constraint $\int_M \xi(g)v_g = 0$, for any $g$ in $\mathcal{M}_\Omega$, implies that the rhs of (4.4.1) obeys the same constraint.

The curvature of $\mathcal{D}$ is easily deduced from (4.4.1), cf. Proposition 4.9.1 below.

### 4.5. Group action of $H(M, J)$ on $\mathcal{M}_\Omega$

The (right) action of the group $H(M, J)$ on $\mathcal{M}_\Omega$ defined in Section 3.1 clearly preserves the Mabuchi metric. For any $X$ in $\mathfrak{h}$ the induced vector field $\tilde{X}$ on $\mathcal{M}_\Omega$ is then a Killing vector field for the Mabuchi metric and we therefore expect that the covariant derivative $\mathcal{D}\tilde{X}$ be skew-symmetric. Recall that $\hat{X}$ is given by $g \in \mathcal{M}_\Omega \mapsto f_g^X$, where $f_g^X$ denotes the real potential of $X$ with respect to $g$. In order to compute $\mathcal{D}\tilde{X}$ we thus need to know the variation of $f_g^X$ and this is given by the following lemma:

**Lemma 4.5.1.** Let $X$ be any vector field in $\mathfrak{h}$. Let $g, \bar{g}$ be any two metrics in $\mathcal{M}_\Omega$, of Kähler forms $\omega$ and $\bar{\omega} = \omega + dd^c\phi$. Let $\xi_g, \xi_{\bar{g}}$ be the dual 1-forms of $X$ with respect to $g, \bar{g}$ respectively, and $\xi_g = \xi_H + df_g^X + d^ch_g^X$, $\xi_{\bar{g}} = \xi_H + df_{\bar{g}}^X + d^ch_{\bar{g}}^X$ their Hodge decomposition, according to Lemma 2.1.1. Then

\begin{align}
\xi_H &= \xi_H, \quad f_g^X = f_{\bar{g}}^X + \mathcal{L}_X\phi, \quad h_g^X = h_{\bar{g}}^X - \mathcal{L}_{JX}\phi.
\end{align}

In particular, the derivative of $f_g^X$ and $h_g^X$ along $\hat{\phi}$ in $T_g\mathcal{M}_\Omega$ is given by:

\begin{align}
f_g^X &= \mathcal{L}_X\hat{\phi}, \quad h_g^X = -\mathcal{L}_{JX}\hat{\phi}.
\end{align}

Equivalently, if $F_g^X = f_g^X + ih_g^X$,

\begin{align}
F_g^X &= 2\mathcal{L}^{1,0}F,
\end{align}

where $X^{1,0} = \frac{1}{2}(X - iJX)$ denotes the $(1,0)$-part of $X$.

**Proof.** We have

\begin{align}
\xi_g &= -JX\omega - JXdd^c\phi = \xi_g - \mathcal{L}_{JX}d^c\phi + d\mathcal{L}_X\phi
&= \xi_g + d\mathcal{L}_X\phi - d\mathcal{L}(J\mathcal{L}_X\phi).
\end{align}

The Hodge decomposition of $\xi_g$ then reads

\begin{align}
\hat{\xi} &= \xi_H + d(f_g^X + \mathcal{L}_X\phi) + d^c(h_g^X - \mathcal{L}_{JX}\phi).
\end{align}

It follows that $\hat{\xi}_H = \xi_H$ — this fits with the fact that the space $\text{harm}$ of harmonic 1-forms is independent of $g$, cf. Section 2.3 — $f_g^X = f_{\bar{g}}^X + \mathcal{L}_X\phi + c_1$, $h_g^X = h_{\bar{g}}^X - \mathcal{L}_{JX}\phi + c_2$, where $c_1 = c_1(X, g, \bar{g}), c_2 = c_2(X, g, \bar{g})$ are constants on $M$, depending on $X, g, \bar{g}$, determined by $\int_M f_g^X v_g = 0$, $\int_M h_g^X v_g = 0$.

By (3.2.4), the first variation of $c_1$ along $\hat{\phi}$ in $T_g\mathcal{M}_\Omega$, when $g$ and $X$ are fixed, is given by

\begin{align}
\hat{c}_1 &= \frac{1}{V_\Omega} \int_M (f_g^X \Delta \hat{\phi} - \mathcal{L}_X\hat{\phi}) v_g = \frac{1}{V_\Omega} \int_M (\hat{\phi}, \Delta f_g^X - \delta \xi_g) = 0,
\end{align}
meaning that $c_1(X, g, \tilde{g})$, as a function of $\tilde{g}$ in $\mathcal{M}_\Omega$, is constant; since $c_1(X, g, g) = 0$, $c_1$ is then 0 everywhere. We similarly show that $c_2 = 0$. We thus get (4.5.1); (4.5.2) follows immediately, as well as (4.5.3).

**Proposition 4.5.1.** For any $X$ in $\mathfrak{h}$ and any $g$ in $\mathcal{M}_\Omega$, the covariant derivative $DX$ at $g$ is given by

\begin{equation}
(4.5.6)

DX = (df, \xi_g - df_g^X) = -\delta^g(f(\xi_g - df_g^X)),
\end{equation}

for any $f$ in $T_g \mathcal{M}_\Omega$, where $\xi_g$ stands for the dual of $X$ with respect to $g$. In particular, $DX$ vanishes at $g$ if and only if $JX$ is a hamiltonian Killing vector field with respect to $g$. Moreover, for any two $f_1, f_2$ in $T_g \mathcal{M}_\Omega$, we have that

\begin{equation}
(4.5.7)

\langle DX_1, f_2 \rangle + \langle DX_2, f_1 \rangle = 0.
\end{equation}

**Proof.** For convenience, we compute $DX$ on $\tilde{\mathcal{M}}_{\omega_0}$. From (4.4.1) and (4.5.2), we infer

\[
DX = f \cdot f_g^X - (df, df_g^X) = \mathcal{L}_X f - (df, df_g^X)_g = (df, \xi_g - df_g^X)_g = -\delta^g(f(\xi_g - df_g^X)),
\]

which is (4.5.6) (for the last equality, we used the fact that $\xi_g - df_g^X = \xi_H + d^c h_g^X$ is co-closed). From the above, we deduce that $DX = 0$ at $g$ if and only if $\xi_g - df_g^X = 0$, hence $X = \text{grad}_g f_g^X$, meaning that $JX = \text{grad}_g f_g^X$ is both hamiltonian and (real) holomorphic, hence a hamiltonian Killing vector field with respect to $g$. Identity (4.5.7) follows readily from (4.5.6). \(\square\)

### 4.6. The equation of geodesics

**Proposition 4.6.1.** Let $\phi(t)$, $t \in [0, T]$, be a (smooth) curve in $\tilde{\mathcal{M}}_{\omega_0}$, for some positive real number $T$ (or $T = +\infty$). Then $\phi(t)$ is a geodesic with respect to the Levi-Civita connection $D$ if and only if $\phi(t)$ satisfies

\begin{equation}
(4.6.1)

\ddot{\phi}(t) - |d\dot{\phi}(t)|_t^2 = 0,
\end{equation}

for all $t$ in $[0, T]$, where $|d\dot{\phi}(t)|_t^2$ denotes the square norm of $d\dot{\phi}(t)$ with respect to the metric $g(t) = \pi(\phi(t))$ for each (fixed) $t$. Similarly, a curve $\omega(t) = \omega_0 + d\hat{d}\phi(t)$ in $\mathcal{M}_\Omega$, with $\mathbb{I}_{\omega_0}(\phi(t)) = 0$ for any $t$ in $[0, T]$, is a geodesic for $D$ if and only if (4.6.1) is satisfied.

**Proof.** The tangent vector field to the curve $\phi(t)$ is $\xi(t) := \dot{\phi}(t)$ at $\phi(t)$ and $\phi(t)$ is a geodesic if and only if $D_{\dot{\phi}(t)} \xi(t) = 0$ for each $t$ in $[0, T]$. In the rhs of (4.4.1), the term $\dot{\xi}(\phi(t))$ is $\dot{\phi}(t)$, so that $D_{\dot{\phi}(t)} \xi(t) = \dot{\phi}(t) - (d\dot{\phi}(t), d\phi(t))_g(t)$. The curve $\phi(t)$ is then a geodesic if and only if (4.6.1) is satisfied. This proves the first assertion. Since the Levi-Civita connection on $\mathcal{M}_\Omega$ coincides with the restriction of $D$ on $\mathbb{I}_{\omega_0}^{-1}(0)$, the second assertion follows. \(\square\)
Remark 4.6.1. For any curve \( \phi(t) \) in \( \mathcal{M}_{\omega_0} \) as above, the first derivative of \( \mathbb{L}_{\omega_0}(\phi(t)) \) is \( \frac{d}{dt} \mathbb{L}_{\omega_0}(\phi(t)) = \tau_{\phi(t)}(\dot{\phi}(t)) = \frac{1}{V_{\Omega}} \int_M \dot{\phi}(t) v_{g(t)}. \) Its second derivative is then
\[
\frac{d^2}{dt^2} \mathbb{L}_{\omega_0}(\phi(t)) = \frac{1}{V_{\Omega}} \int_M (\dddot{\phi}(t) - \dot{\phi}(t) \Delta_{g(t)} \dot{\phi}(t)) v_{g(t)}(t)
\]
(4.6.2)

If \( \phi(t) \) is a geodesic, \( \mathbb{L}_{\omega_0}(\phi(t)) \) is then an affine function of \( t \). In particular, \( \mathbb{L}_{\omega_0}(\phi(t)) \equiv 0 \) whenever \( \phi(t) \) is a geodesic, \( \phi(0) \) belongs to \( \mathbb{L}_{\omega_0}^{-1}(0) \) and \( \dot{\phi}(0) \) belongs to \( T_{\phi(0)}\mathbb{L}_{\omega_0}^{-1}(0) \). This confirms that \( \mathbb{L}_{\omega_0}^{-1}(0) \) is totally geodesic in \( \mathcal{M}_{\omega_0} \).

The equation of geodesics admits various alternative formulations. One of them relies on the following general construction. For any chosen basepoint \( \omega_0 \) and any element \( \omega = \omega_0 + \frac{d}{dt} \phi \) in \( \mathcal{M}_{\Omega} \) let \( c(t) = \omega + \frac{d}{dt} \phi \) be any curve in \( \mathcal{M}_{\Omega} \) joining \( \omega_0 \) to \( \omega = \omega_1 \), determined by the curve \( \phi(t) = \phi_t \) in \( \mathbb{L}_{\omega_0}^{-1}(0) \subset \mathcal{M}_{\omega_0} \). Denote by \( X_t \) the family of vector fields defined by

\[
X_t = -\text{grad}_{g_t} \dot{\phi}(t),
\]
where \( g_t \) stands for the riemannian metric associated to \( \omega_t \). We denote by \( \Phi_t \) the corresponding “flow”, i.e. the 1-parameter family of diffeomorphisms of \( M \) determined by

\[
\frac{d}{dt} \Phi_t(x) = X_t(\Phi_t(x)),
\]
(4.6.4)
\[\Phi_0 = \text{Id},\]

for each \( t \) in \([0,1]\) and each \( x \) in \( M \) (where \( \text{Id} \) stands for the identity map). If \( Z := X_t + \partial / \partial t \) is viewed as a vector field on the product \( M \times [0,1] \), and if \( \Phi^Z_t \) denotes the flow of \( Z \), then \( \Phi_t \) is related to \( \Phi^Z_t(x,0) \) by \( \Phi^Z_t((x,0)) = (\Phi_t(x), t) \).

Notice that \( \Phi_t \) is not a flow in the usual sense; in particular, \( \Phi_{t_1 + t_2} \neq \Phi_{t_1} \circ \Phi_{t_2} \) and \( \Phi_{-t} \neq \Phi^{-1}_t \) in general. We have instead

Lemma 4.6.1. Let \( \Phi_t \) the flow of \( X_t \) defined by (4.6.4). Then:

(i) \( \Phi^{-1}_t \) is the flow of the 1-parameter vector field \(-\Phi^{-1}_t \cdot X_t \), where, for each \( t \), \( \Phi^{-1}_t \cdot X_t \) denotes the direct image of the vector field \( X_t \), cf. Section 1.4.

(ii) For any tensor field \( T \) and for any fixed \( t \), the Lie derivative \( \mathcal{L}_{X_t} T \) of \( T \) along the vector field \( X_t \) is given by:

\[
\mathcal{L}_{X_t} T = \Phi_t \cdot \frac{d}{dt}(\Phi^{-1}_t \cdot T).
\]
(4.6.5)

Proof. (i) From \( \frac{d}{dt}(\Phi^{-1}_t \circ \Phi_t) = 0 \), we infer: \( \frac{d}{dt} \Phi^{-1}_t(\Phi_t(x)) = -((\Phi^{-1}_t)_* X_t(\Phi_t(x))) = -((\Phi^{-1}_t \cdot X_t)(\Phi^{-1}_t(x)), \) for any \( x \) in \( M \). By substituting \( \Phi^{-1}_t(x) \), we get \( \frac{d}{dt} \Phi^{-1}_t(x) = -((\Phi^{-1}_t \cdot X_t)(\Phi^{-1}_t(x)), \) as required.

(ii) The derivative \( \frac{d}{dt}(\Phi^{-1}_t \cdot T) \) at \( t \) is better written as \( \frac{d}{dt}_{|t=0} \Phi^{-1}_t \cdot T \).

From (i), we infer \( \frac{d}{dt}_{|t=0} (\Phi_t \circ \Phi^{-1}_{t+s}) = 0 \) for each \( t \). From the definition (1.4.1) of the Lie derivative, we then get: \( \frac{d}{dt}_{|t=0} \Phi^{-1}_t \cdot T = \frac{d}{dt}_{|t=0} (\Phi^{-1}_t \circ \Phi_t \circ \Phi^{-1}_t \cdot T = \Phi^{-1}_t \cdot \frac{d}{dt}_{|t=0} (\Phi_t \circ \Phi^{-1}_t \cdot T = \Phi^{-1}_t \cdot \mathcal{L}_{X_t} T, \) which is (4.6.5). \( \square \)
Since $\Phi_t^* \omega_t = \Phi_t^{-1} \cdot \omega_t$, from (4.6.5), we infer
\begin{equation}
\frac{d}{dt}(\Phi_t^* \omega_t) = \Phi_t^*(\mathcal{L}_t \omega_t + \dot{\omega}_t)
= \Phi_t^*(d(X_t \omega_t) + \dot{\omega}_t)
= \Phi_t^*(-dd^c \dot{\phi}_t + dd^c \dot{\phi}_t) = 0.
\end{equation}

We thus get
\begin{equation}
\Phi_t^* \omega_t = \omega_0,
\end{equation}
for each value of $t$ in $[0, 1]$. This can be viewed as a special and simple case of the Moser lemma, cf. [74]. Accordingly, the curve $t \mapsto \Phi_t$ will be called the Moser lift determined by $\dot{c}$ in the identity component $\text{Diff}_0(M)$ of the group of diffeomorphisms of $M$ (more on the Moser lift in Section 4.7.)

By using the Moser lift, we get the following characterization of geodesics in $\mathcal{M}_\Omega$ or $\mathcal{M}_\omega$:

**Proposition 4.6.2. For any curve $\omega_t = \omega_0 + dd^c \dot{\phi}_t$, $t \in [0, 1]$, in $\mathcal{M}_\Omega$, denote by $\Phi_t$ the corresponding Moser lift in $\text{Diff}_0(M)$. Then, $\omega_t$ is a geodesic if and only if**

\begin{equation}
\Phi_t^* \dot{\phi}_t = \dot{\phi}_0,
\end{equation}
for each $t$ in $[0, 1]$. Equivalently,

\begin{equation}
\dot{\phi}_t = \int_0^t (\dot{\phi}_0 \circ \Phi_t^{-1}) \, ds.
\end{equation}

**Proof.** Assume that $\omega_t$ is a geodesic in $\mathcal{M}_\Omega$. Then:

\begin{equation}
\frac{d}{dt}(\Phi_t^* \dot{\phi}_t) = \Phi_t^*(\mathcal{L}_t \dot{\phi}_t + \ddot{\phi}_t) = \Phi_t^*(-(d\dot{\phi}_t, d\dot{\phi}_t)_g + \dddot{\phi}_t) = 0.
\end{equation}

We then get (4.6.8), hence (4.6.9). Conversely, by differentiating both sides of (4.6.8) we get $0 = \Phi_t^*(\dot{\phi}_t + \mathcal{L}_t \dot{\phi}_t)$, with $X_t$ defined by (4.6.3), hence $\tilde{\phi}_t - (d\dot{\phi}_t, d\dot{\phi}_t)_g = 0$, where $g_t$ denotes the metric associated to $\omega_t$. \qed

**Proposition 4.6.3.** (i) Let $X$ be any (real) holomorphic vector field in $\mathfrak{h}_{\text{red}}$ of the form $X = \text{grad}_{g_0} f_{g_0}^X$, for some metric $g_0$ in $\mathcal{M}_\Omega$, of Kähler form $\omega_0$, and for some real function $f_{g_0}^X$ satisfying $\int_M f_{g_0}^X \nu_{g_0} = 0$. Denote by $\Phi_t^X$ the flow of $X$. Then, the curve $(g_t, \omega_t)$ in $\mathcal{M}_\Omega$ defined by $(g_t = (\Phi_t^X)^* g_0, \omega_t = (\Phi_t^X)^* \omega_0)$ is a geodesic. Moreover, if $\omega_t$ is written as $\omega_t = \omega_0 + dd^c \dot{\phi}_t$, with $\mathbb{L}_{\omega_0}(\dot{\phi}_t) = 0$, then $X = \text{grad}_{g_t} \dot{\phi}_t$, so that $\dot{\phi}_t$ is a Killing potential with respect to $g_t$ for any $t$, and $\Phi_t = \Phi_t^X$ is then the Moser lift of the geodesic $(g_t, \omega_t)$.

(ii) Conversely, let $(g_t, \omega_t = \omega_0 + dd^c \dot{\phi}_t)$ be a (smooth) curve in $\mathcal{M}_\Omega$ starting from $\omega_0$, such that $\dot{\phi}_t$ is a Killing potential with respect to $g_t$ for each $t$. Then this curve is contained in the $H_0(M, J)$-orbit of $\omega_0$. Moreover, this curve is a geodesic in $\mathcal{M}_\Omega$ if and only if the 1-parameter (real) vector field $\text{grad}_{g_t} \dot{\phi}_t$ is independent of $t$. If so, $\text{grad}_{g_t} \dot{\phi}_t$ is a well-defined element $X$ in $\mathfrak{h}_{\text{red}}(M, J)$ and we have $\omega_t = (\Phi_t^X)^* \omega_0$, where $\Phi_t^X$ denotes the flow of $X$.

**Proof.** (i) First notice that $\omega_t = \Phi_t^* \omega_0$ lives in $\mathcal{M}_\Omega$ for each $t$, as $\Phi_t$ belongs to $H_{\text{red}}(M, J)$, hence preserves the complex structure $J$ and
the Kähler class $\Omega$. We then have $\omega_t = \omega_0 + dd^c \phi_t$, where, we assume, $L_{\omega_0}(\phi_t) = 0$, hence $\omega_t = dd^c \phi_t$, with $\int_M \phi_t v_g = 0$. From $\omega_t = \Phi_t^* \omega_0$, we infer $\dot{\omega}_t = \Phi_t^* (L_X \omega_0) = \Phi_t^* dd^c f_{X_0} = dd^c (\Phi_t^* f_X)$, with $\int_M \Phi_t^* f_X^v g_t = \int_M \Phi_t^* f_{X_0}^v g_0 = 0$. We thus get: $\dot{\phi}_t = \Phi_t^* f_{X_0}$ for any $t$. On the other hand, $X = \Phi_t^{t^1} \cdot X = \Phi_t^{-1} \cdot \text{grad}_{g_t} f_{X_0}^g \Phi_t (\Phi_t^* f_X)$, for any $t$; we infer $\phi_t^X = \Phi_t^* f_{X_0} = \dot{\phi}_t$, so that $X = \text{grad}_{g_t} \phi_t$ for each $t$; this means that $\Phi_t^{-1}$ is the Moser lift of the curve $\omega_t$, and $\Phi_t^* \phi_0 = \dot{\phi}_t$. By Proposition 4.6.2, this implies that the curve $\omega_t$ is a geodesic.

(ii) Define $X_t = - \text{grad}_{g_t} \phi_t$ and let $\Phi_t$ be the “flow” of the 1-parameter vector field $X_t$, as defined above, so that $\Phi_t^* \omega_t = \omega_0$ for any $t$, see (4.6.5). Since $\dot{\phi}_t$ is a Killing potential, $X_t$ is a (real) holomorphic vector field in $\mathfrak h_{\text{red}}$, by Proposition 2.2.1. By using (4.6.5), we infer that $\Phi_t^{-1} \cdot J$ is constant along the curve, hence that $\Phi_t^{-1} \cdot J = J$, meaning that $\Phi_t^{-1}$ belongs to $H(M)$ for any $t$. Moreover, since $\frac{d}{dt} \Phi_t \circ \Phi_t^{-1} = X_t$ belongs to $\mathfrak h_{\text{red}}$, we conclude that $\Phi_t$ actually belongs to $H_{\text{red}}(M,J)$ for each $t$.

By using (3.3) in the proof of Proposition 3.3.1, we get $X_t = \text{grad}_{g_t} \phi_t + \text{grad}_{g_t} \phi_t = (J X_t, dd^c \phi_t)^{\text{red}} + \text{grad}_{g_t} \phi_t = (L_{J X_t} dd^c \phi_t)^{\text{red}}$ - $\text{grad}_{g_t} (J X_t, dd^c \phi_t) + \text{grad}_{g_t} \phi_t$. Since $X_t$, hence also $J X_t$, belongs to $\mathfrak h_{\text{red}}$, we have $L_{J X_t} dd^c \phi_t = dd^c (\phi_t (J X_t)) = 0$, as $\phi_t (J X_t) = (d \phi_t, J d \phi_t)_{g_t} = 0$. We thus eventually get: $X_t = \text{grad}_{g_t} (X_t, dd^c \phi_t, \phi_t) = 0$, meaning that $\dot{X}_t = 0$ if and only if the curve $\omega_t = \omega_0 + dd^c \phi_t$ is a geodesic in $\mathcal{M}_\Omega$. If so, $X = \text{grad}_{g_0} \phi_t$ is a well-defined vector field in $\mathfrak h_{\text{red}}(M,J)$, whose flow is $\Phi_t^X = \Phi_t^{-1}$, and $\omega_t = (\Phi_t^X)^* \omega_0$ for each $t$.

REMARK 4.6.2. Assume that $(M,J)$ is a (compact) Riemann surface, i.e., that $m = 1$. Fix any Kähler form $\omega_0$, corresponding to a Kähler metric $g_0$ (note that $\omega_0$ coincides with the volume form $v_g$) and denote by $\Omega = [\omega_0]$ its Kähler class. Any curve in $\mathcal{M}_\Omega$ can be written as:

$$\omega_t = \omega_0 + dd^c \phi_t = (1 - \Delta \phi_t) \omega_0.$$  \hfill (4.6.11)

On the other hand, for any (real) function $\phi$, we have that $|d \phi|_{g_0}^2 \omega_t = |d \phi|_{g_0}^2 \omega_0 = d \phi \wedge d^c \phi$, and similarly $dd^c \phi = - \Delta_{g_0} \phi \omega_t = - \Delta_{g_0} \phi \omega_0$, for each value of $t$. The geodesic equation can then be written in the following form:

$$\ddot{\phi}_t (1 - \Delta_{g_0} \phi_t) - |d \phi_t|_{g_0}^2 = 0,$$  \hfill (4.6.12)

where the laplacian and the norm are relative to the fixed chosen metric $g_0$.

4.7. The Moser lift as a horizontal lift

As observed by S. Donaldson in [74], part of the discussion in Section 4.6 can be reinterpreted in terms of a connection on a $\text{Sp}_0(M,\omega_0)$-principal bundle over $\mathcal{M}_\Omega$ as follows, where $\text{Sp}_0(M,\omega_0)$ denotes the identity component of the group of symplectomorphisms of $(M,\omega_0)$.

Denote by $\mathcal{Q}$ the space of pairs $(\Phi, \omega)$ in $\text{Diff}_0(M) \times \mathcal{M}_\Omega$ such that

$$\omega = \Phi \cdot \omega_0.$$  \hfill (4.7.1)
The natural projection to \( \mathcal{M}_\Omega \) then makes \( \mathcal{Q} \) into a \( \text{Sp}_0(M,\omega_0) \)-principal bundle over \( \mathcal{M}_\Omega \), via the (right) action \((\Phi,\omega) \cdot \gamma = (\Phi \circ \gamma,\omega)\), for any \((\Phi,\omega)\) in \( \mathcal{Q} \) and any \( \gamma \) in \( \text{Sp}_0(M,\omega_0) \).

Fix \((\Phi,\omega)\) in \( \mathcal{Q} \) and let \((\Phi_t,\omega(t))\) be any curve in \( \mathcal{Q} \), with \( \Phi_0 = \Phi \) and \( \omega(0) = \omega \). Then, \( \omega(t) = \Phi_t \cdot \omega_0 = \Phi_t \Phi^{-1} \cdot \omega \). It follows that \( \dot{\omega}(0) = -\mathcal{L}_\chi \omega = -d\langle \iota_X \omega, \cdot \rangle \), whereas \( \dot{\omega}(0) = d\iota_X f \), with \( \int_M f \, \eta_g = 0 \), where, as usual, \( \eta = \omega(\cdot,J) : f \) is then \( \omega(0) \) as an element of \( T_{\omega_0}\mathcal{M}_\Omega \) according to our usual identification. The tangent space \( T_{(\Phi,\omega)}\mathcal{Q} \) is then naturally identified with the space of pairs \((X,f)\), where \( X \) is a (real) vector field and \( f \) a (real) function, with \( \int_M f \, \eta_g = 0 \), such that the (real) 1-form \( \iota_X \omega + d^c f \) is closed.

Recall that the Lie algebra \( \mathfrak{sp}(M,\omega_0) \) of \( \text{Sp}_0(M,\omega_0) \) is the space of (real) vector fields \( Z \) on \( M \) such that \( \mathcal{L}_Z \omega = 0 \); equivalently, the \( \omega_0 \)-symplectic dual 1-form \( \iota_X \omega \) is closed.

Let \( Z \) be any element of \( \mathfrak{sp}(M,\omega_0) \), generated by a curve \( \gamma_t \in \text{Sp}_0(M,\omega_0) \), with \( \gamma_0 = 1 \) (e.g. \( \gamma_t = \Phi_t^Z \), the flow of \( Z \)). The induced vertical vector \( \dot{Z}(\Phi,\omega) \) on \( \mathcal{Q} \) at \((\Phi,\omega)\), tangent to the curve \((\Phi_t = \Phi \circ \gamma_t,\omega)\) at \( t = 0 \), is then identified with the pair \((X = \Phi \cdot Z,0)\).

In view of this, we define a \( \text{Sp}_0(M,\omega_0) \)-equivariant connection 1-form, \( \eta \), on \( \mathcal{Q} \) in the following way: for any \((X,f)\) in \( T_{(\Phi,\omega)}\mathcal{Q} \), \( \eta((X,f)) \) is the \( \omega_0 \)-symplectic dual vector field of the (closed) 1-form \( \Phi^*(\iota_X \omega + d^c f) \).

We readily check that \( \dot{\eta}(\dot{Z}(\Phi,\omega)) = Z \) for any \( Z \) in \( \mathfrak{sp}(M,\omega_0) \) and that \( \eta \) is \( \text{Sp}_0(M,\omega_0) \)-equivariant as required.

**Lemma 4.7.1.** For any \((g,\omega)\) in \( \mathcal{M}_\Omega \), for any \( f \) in \( T_g \mathcal{M}_\omega \) and for any \((\Phi,\omega)\) in \( \mathcal{Q} \) over \((g,\omega)\), the horizontal lift of \( f \) in \( T_{(\Phi,\omega)}\mathcal{Q} \) is \((-\text{grad}_g \Phi, f)\).

**Proof.** Any lift of \( f \) in \( T_{(\Phi,\omega)}\mathcal{Q} \) is of the form \((X,f)\), with \( \iota_X \omega + d^c f \) closed. Such a lift is horizontal if \( \eta((X,f)) = 0 \). By the above this means that \( \Phi^*(\iota_X \omega + d^c f) = 0 \), hence \( \iota_X \omega + d^c f = 0 \); equivalently, \( X = -\text{grad}_g \Phi \).

**Proposition 4.7.1.** For any curve \( \omega_t = \omega_0 + d\phi_t \) in \( \mathcal{M}_\Omega \) starting from \( \omega_0 \), a curve \( \Phi_t \) in \( \text{Diff}_0(M) \), with \( \Phi_0 = \text{Id} \), is the Moser curve of \( \omega_t \) if and only if \((\Phi_t,\dot{\omega}_t)\) is a horizontal lift of \( \dot{\omega}_t \) in \( \mathcal{Q} \) with respect to the connection defined by \( \eta \).

**Proof.** Let \((\Phi_t,\dot{\omega}_t)\) be any lift of \( \omega_t \) in \( \mathcal{Q} \), with \( \Phi_0 = \text{Id} \). By definition of \( \mathcal{Q} \), \( \Phi_t^* \omega_t = \omega_0 \) for all \( t \). Moreover, the tangent vector field of the curve \((\Phi_t,\omega_t)\) is the pair \((X_t,\dot{\omega}_t)\), where \( X_t \) is the time dependent vector field defined by (4.6.4). By Lemma 4.7.1, \((X_t,\dot{\omega}_t)\) is horizontal if and only if \( X_t = -\text{grad}_g \Phi_t \); this is the way the Moser lift was defined.

Devote by \( C^\infty(M,\omega_0) \) the space of (smooth) real functions on \( M \) such that \( \int_M f \, \eta_g = 0 \) \( (C^\infty(M,\omega_0)) \) is then the tangent space \( T_{\omega_0}\mathcal{M}_\Omega \). We regard \( C^\infty(M,\omega_0) \) as a representation space of \( \text{Sp}_0(M,\omega_0) \) and we consider the induced vector bundle \( \mathcal{Q} \times_{\text{Sp}_0(M,\omega_0)} C^\infty(M,\omega_0) \) over \( \mathcal{M}_\Omega \).

**Proposition 4.7.2.** (i) We have a natural identification of vector bundles:

\[
T\mathcal{M}_\Omega = \mathcal{Q} \times_{\text{Sp}_0(M,\omega_0)} C^\infty(M,\omega_0)
\]
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\[(4.7.3) \quad [(\Phi, \omega), f] \mapsto \Phi \cdot f.\]

(ii) In this identification, the linear connection, \(\mathcal{D}^\eta\), on \(T\mathcal{M}_\Omega\) induced by \(\eta\) coincides with the Levi-Civita connection \(\mathcal{D}\) of the Mabuchi metric.

Proof. (i) To any element of \(Q \times \text{Sp}(M, \omega_0) C^\infty(M, \omega_0)\) represent by \((\Phi, \omega, f)\) in \(Q \times C^\infty(M, \omega_0)\) we associate \(\Phi \cdot f\) in \(T\mathcal{M}_\Omega\) (notice that \(\int_M (\Phi \cdot f) v_g = \int_M \Phi \cdot (fv_g) = \int_M f v_g = 0\)); if \((\Phi \circ \gamma, \omega), \gamma^{-1} \cdot f)\) is another representative, with \(\gamma\) in \(\text{Sp}(M, \omega_0)\), we clearly get the same element of \(T\mathcal{M}_\Omega\).

(ii) It suffices to check that \(\mathcal{D}^\eta_1 \tilde{f}_2 = \mathcal{D}_{f_1} \tilde{f}_2\) is satisfied for any \(f_1\) in \(T\mathcal{M}_\Omega\) and for any “constant” vector field \(\tilde{f}_2 = f_2 - \frac{1}{V_\Omega} \int_M f_2 v_g\). Without loss of generality, we can choose \(g = g_0\). Via \((4.7.2), \tilde{f}_2\) is represented by \(\left((\Phi, \omega), \Phi^{-1} \cdot f_2 - \frac{1}{V_\Omega} \int_M f_2 \frac{\omega^m}{m!}\right)\), whereas \(f_1\) is determined by the curve \(\omega_t = \omega_0 + d\omega \phi_t\), with \(\phi_0 = f_1\). By choosing \((\Phi_t, \omega_t)\) the horizontal lift of \(\omega_t\) in \(Q\) starting from \((\Phi_0 = \text{Id}, \omega_0)\), the covariant derivative with respect to the connection \(\eta\) of \(f_2\) is simply given by

\[(4.7.4) \quad \mathcal{D}^\eta_1 \tilde{f}_2 = ((\text{Id}, \omega_0), \frac{df_1}{dt}_{t=0} (\Phi_t^{-1} \cdot f_2 - \frac{1}{V_\Omega} \int_M f_2 \frac{\omega^m}{m!}).\]

By using \(\frac{df_1}{dt}_{t=0} = -\text{grad}_{g_0} f_1\), given by Lemma 4.7.1, and \((3.2.4), we thus get \n
\[(4.7.5) \quad \mathcal{D}^\eta_1 \tilde{f}_2 = ((\text{Id}, \omega_0), -\text{grad}_{g_0} f_1, \text{grad}_{g_0} f_2) + \frac{1}{V_\Omega} \int_M (\text{grad}_{g_0} f_1, \text{grad}_{g_0} f_2) v_{g_0}).\]

This is \(\mathcal{D}_1 \tilde{f}_2\), as given by \((4.4.1), in the identification \((4.7.2).\)

Remark 4.7.1. Let \(\omega_t = \omega_0 + d\omega \phi_t\) be a curve in \(\mathcal{M}_\Omega\) starting from \(\omega_0\). Let \(\Phi_t\) the corresponding Moser lift in \(\text{Diff}_0(M)\), so that \((\Phi_t, \omega_t)\) is a horizontal lift of \(\omega_t\) in \(Q\) by Lemma 4.7.1. The tangent vector field along the curve \(\omega_t\) is \(\phi_t\). By \((4.7.3), this corresponds to the curve \([((\Phi_t, \omega_t), \Phi_t^* \phi_t)\) in \(Q \times \text{Sp}(M, \omega_0) C^\infty(M, \omega_0)\). Since \((\Phi_t, \omega_t)\) is horizontal in \(Q\), this curve is \(\eta\)-horizontal in \(Q \times \text{Sp}(M, \omega_0) C^\infty(M, \omega_0)\) for the induced connection if and only if \(\Phi_t^* \phi_t\) is constant, equal to \(\phi_0\). By using Proposition 4.7.2, we thus recover the characterization of geodesics of the Mabuchi space \(\mathcal{M}_\Omega\) given by Proposition 4.6.2.

4.8. The geodesic equation as a Monge-Ampère equation

As shown by S. Semmes in [174], the geodesic equation \((4.6.1) can be written as a (degenerate) Monge-Ampère equation in the following way. Let \(\phi_t\) be a geodesic in \(\mathcal{M}_\omega\), where \(t\) runs from 0 to \(T\). Then, \(\phi_t\) can be viewed as a real function defined on the product \(M \times [0, T]\). By introducing a new real variable \(s\), we define a function \(\Phi = \Phi(x, t, s)\) on the product \(\tilde{M} := M \times [0, T] \times S^1\) by putting \(\Phi(x, t, s) = \phi_t(x)\), for any \(x\) in \(M\). The additional factor \(\Sigma := [0, T] \times S^1\) is then made into a Riemann surface via the complex structure \(J_2\) defined by \(J_2 \partial/\partial t = \partial/\partial s, or d't = ds\). By combining \(J_2\) with \(J\) we get a complex structure on \(\tilde{M}\), whose twisted differential is
still denoted by $d^c$. Similarly, $\omega_0$ is viewed a 2-form defined on $\tilde{M}$ (but of course $\omega_0$ is no longer a symplectic form on $\tilde{M}$, as $\omega_0^{m+1} = 0$). We then have [174]:

**Proposition 4.8.1.** A curve $\phi_t$ in $\tilde{M}_{\omega_0}$ satisfies the geodesic equation (4.6.1) if and only if $\Phi$ satisfies the following Monge-Ampère equation:

\[(\omega_0 + dd^c \Phi)^{m+1} = 0,\]

\[\text{on the complex manifold } \tilde{M}.\]

**Proof.** We readily get $d\Phi = d\dot{\phi}_t + \ddot{\phi}_t dt$, hence $d^c\Phi = d^c \dot{\phi}_t + \dot{\phi}_t ds$ and

\[dd^c\Phi = dd^c \dot{\phi}_t - d^c \ddot{\phi}_t \wedge dt + \dot{\phi}_t \wedge ds + \dot{\phi}_t dt \wedge ds.\]

We then have $\omega_0 + dd^c \Phi = \omega_t + \psi_t$, with $\omega_t = \omega_0 + dd^c \phi_t$ and $\psi_t = -d^c \ddot{\phi}_t \wedge dt + d\dot{\phi}_t \wedge ds + \dot{\phi}_t dt \wedge ds$. It follows that

\[(\omega_0 + dd^c \Phi)^{m+1} = (m + 1)(\ddot{\phi}_t \omega_t^m - m\omega_t^{m-1} \wedge \dot{\phi}_t \wedge d^c \phi_t) \wedge dt \wedge ds = (m + 1)(\ddot{\phi}_t - (d\dot{\phi}_t, d\dot{\phi}_t)) \omega_t^m \wedge dt \wedge ds.\]

The above computation is performed for $S^1$-invariant solutions $\Phi$ of (4.8.1). More generally, we have (cf. [74] Formula (29)):

**Proposition 4.8.2.** Let $\Phi$ be a (smooth) real function on $M \times [0,T] \times S^1$, viewed as a family $\{\varphi_{t,s}\}$ of functions on $M$, parametrized by $(t,s)$ in $[0,T] \times S^1$. Assume that $\varphi_{t,s}$ is the relative Kähler potential with respect to $\omega_0$ of an element $(g_{t,s}, \omega_{t,s})$ of $M_{\Omega}$ for any $(t,s)$ in $[0,T] \times S^1$. Then $\Phi$ is a solution to (4.8.1) if and only if the family $\varphi_{t,s}$ satisfies:

\[\varphi_{tt} + \varphi_{ss} - |d\varphi_t - d^c \varphi_s|^2_{g_{t,s}} = 0,\]

where $\varphi_t, \varphi_{tt}$ denote the first and second derivative of $\varphi = \varphi_{t,s}$ with respect to $t$, $\varphi_s, \varphi_{ss}$ the first and second derivative of $\varphi = \varphi_{t,s}$ with respect to $s$.

**Proof.** As in the proof of Proposition 4.8.1, we have: $d^c \Phi = d^c \varphi_{t,s} + \dot{\varphi}_t ds - \ddot{\varphi}_t dt$, hence $dd^c \Phi = dd^c \varphi_{t,s} - d^c \ddot{\varphi}_t \wedge dt - d^c \dot{\varphi}_t \wedge ds + \dot{\varphi}_t dt \wedge ds - \ddot{\varphi}_s ds \wedge dt + d\dot{\varphi}_t \wedge ds - d\dot{\varphi}_s \wedge dt$, so that: $\omega_0 + dd^c \Phi = \omega_{t,s} + \psi$ with

\[\psi = (\varphi_{tt} + \varphi_{ss}) dt \wedge ds - (d^c \dot{\varphi}_t + d\dot{\varphi}_s) \wedge dt + (d^c \ddot{\varphi}_t - d\ddot{\varphi}_s) \wedge ds.\]

In particular,

\[\psi^2 = 2(d^c \dot{\varphi}_t + d\dot{\varphi}_s) \wedge (d\dot{\varphi}_t - d^c \dot{\varphi}_s) \wedge dt \wedge ds\]

and $\psi^3 = 0$. It follows that

\[(\omega_0 + dd^c \Phi)^{m+1} = (\omega_{t,s} + \psi)^{m+1} = (m + 1)\omega_{t,s}^m \wedge \psi + \frac{(m + 1)m}{2} \omega_{t,s}^{m-1} \wedge \psi^2 = (m + 1)(\varphi_{tt} + \varphi_{ss} - |d\varphi_t - d^c \varphi_s|^2) \omega_{t,s}^m \wedge dt \wedge ds.\]

The above Monge-Ampère approach has proved an effective way of studying geodesics in $\tilde{M}_{\omega_0}$ and $M_{\Omega}$, in particular in [55], [47], [78], [57], [59], [60], [56], cf. Section 4.15 below.
4.9. The curvature of the Mabuchi metric

**Proposition 4.9.1.** The curvature, $R^D$, of the Levi-Civita connection $D$ of the Mabuchi metric on $\mathcal{M}_{\omega_0}$ is given at $\phi$ by

$$R^D_{f_1,f_2,f_3} = -\{\{f_1,f_2\}_\omega,f_3\}_\omega,$$

for any $f_1, f_2, f_3$ in $T_\phi\mathcal{M}_{\omega_0} = C^\infty(M,\mathbb{R})$, where $\{\cdot,\cdot\}_\omega$ denotes the Poisson bracket with respect to $\omega = \omega_0 + df^\phi$.

The same formula holds for the curvature of $D$ on $\mathcal{M}_\Omega$ for any $(g, \omega)$ in $\mathcal{M}_\Omega$ and any $f_1, f_2, f_3$ in $T_g\mathcal{M}_\Omega = C^\infty_{0,g}(M, \mathbb{R})$.

**Proof.** According to the general definition of the curvature, cf. (1.18.1), $R^D$ has the following expression:

$$R^D_{f_1,f_2,f_3} = -\mathcal{D}_{f_1}(\mathcal{D}_{f_2}f_3) + \mathcal{D}_{f_2}(\mathcal{D}_{f_1}f_3),$$

for any $f_1, f_2, f_3$ in $T_\phi\mathcal{M}_\Omega$, regarded as constant vector fields on $\mathcal{M}_\Omega$ in the rhs (with the effect that $\mathcal{D}_{[f_1,f_2]}f_3 = 0$.) From (4.4.1), we readily infer

$$-\mathcal{D}_{f_1}(\mathcal{D}_{f_2}f_3) = f_1 \cdot (df_2, df_3)_g - (df_1, d((df_2, df_3)_g))_g,$$

with $g = \pi(\phi)$. By writing $(df_2, df_3)_g = \Lambda_g(df_2 \wedge d^c f_3)$ and by using (3.2.10), we get $f_1 \cdot (df_2, df_3)_g = \Lambda_g(f_1)(df_2 \wedge d^c f_3) = -(dd^c f_1, df_2 \wedge d^c f_3)$. In terms of the Levi-Civita connection $D = D^g$ of $g$, this can be rewritten

$$f_1 \cdot (df_2, df_3)_g = -(D_{\text{grad}_g f_2} d^c f_1)(\text{grad}_g f_3) + (D_{\text{grad}_g f_3} d^c f_1)(\text{grad}_g f_2)$$

whereas $(df_1, d((df_2, df_3)_g))_g$ can be rewritten

$$(df_1, d((df_2, df_3)_g))_g = (D_{\text{grad}_g f_1} df_2)(\text{grad}_g f_3) + (D_{\text{grad}_g f_3} df_2)(\text{grad}_g f_2).$$

Taking into account the symmetry of the hessian $Ddf$, most terms in the rhs of (4.9.2) cancel, and (4.9.2) then becomes:

$$R^D_{f_1,f_2,f_3} = D_{\text{grad}_g f_3} d^c f_1(\text{grad}_g f_2) - D_{\text{grad}_g f_2} d^c f_1(\text{grad}_g f_3).$$

In terms of symplectic gradients and of Poisson brackets with respect to the Kähler form $\omega = \omega_0 + dd^c \phi$, cf. (1.2.10), this can be rewritten

$$R^D_{f_1,f_2,f_3} = -D_{\text{grad}_g f_2} df_1(\text{grad}_g f_2) + D_{\text{grad}_g f_3} df_2(\text{grad}_g f_1)$$

$$= -d\omega(D_{\text{grad}_g f_1} \text{grad}_g f_2 - D_{\text{grad}_g f_3} \text{grad}_g f_1, \text{grad}_g f_3)$$

$$(4.9.7) = -\omega([\text{grad}_g f_1, \text{grad}_g f_2], \text{grad}_g f_3)$$

$$(4.9.7) = -\omega(\text{grad}_g(\{f_1, f_2\}_\omega), \text{grad}_g f_3)$$

$$(4.9.7) = -\{f_1, f_2\}_\omega \cdot f_3\omega,$$

which is (4.9.1). The second assertion follows immediately (notice that the rhs of (4.9.1) only depends of $f_1, f_2, f_3$ modulo constants and is of mean value 0 with respect to $g$, as are all Poisson brackets).
In particular, the sectional curvature of \( \tilde{g}_\Omega \) and \( g_\Omega \) is everywhere non-positive, \([142],[74]\).

### 4.10. The Mabuchi K-energy

For any \( g \in \mathcal{M}_\Omega \), the difference \( s_g - \bar{s} \), where, as usual, \( s_g \) denotes the scalar curvature of \( g \) and \( \bar{s} = \frac{\bar{S}_\Omega}{V_\Omega} \) its mean value (the same for all \( g \)'s in \( \mathcal{M}_\Omega \)), can be viewed as an element of \( T_g \mathcal{M}_\omega \) and the assignment \( g \mapsto s_g - \bar{s} \) can therefore be regarded as a vector field on \( \mathcal{M}_\Omega \), denoted by \( \hat{s} \). Similarly, the assignment, \( g \mapsto s_g v_g \), can be viewed as a 1-form on \( \mathcal{M}_\Omega \). This 1-form will be denoted by \( \sigma \) and called the **scalar curvature 1-form**.

The vector field \( \hat{s} \) and the 1-form \( \sigma \) can be considered as a riemannian dual pair on \( \mathcal{M}_\Omega \) in the following (weak) sense:

\[
\sigma \phi(f) = \langle \hat{s}, f \rangle_g,
\]

for each \( g \in \mathcal{M}_\Omega \) and each \( f \) in \( C^\infty_0(M, \mathbb{R}) = T_g \mathcal{M}_\Omega \). In particular, the Calabi functional can be rewritten

\[
C(\phi) = \sigma(\hat{s}) + \frac{\bar{S}_\Omega}{V_\Omega}.
\]

Both \( \sigma \) and \( \hat{s}_0 \) are \( H(M, J) \)-invariant, as \( s_{\gamma^* g} = \gamma^*(s_g) \) and \( v_{\gamma^* g} = \gamma^*(v_g) \) for each riemannian metric \( g \) on \( M \) and each diffeomorphism \( \gamma \) of \( M \).

A basic observation is that the the covariant derivative of \( \sigma \) with respect to \( D \) essentially coincides with the fourth order Lichnerowicz operator \( L \) introduced in Section 1.23 in the following precise sense:

**Lemma 4.10.1.** The covariant derivative of \( \hat{s} \) and of \( \sigma \) with respect to \( D \) are given by

\[
D_f \hat{s} = -2L(f),
\]

and by

\[
D_f \sigma = -2L(f) v_g
\]

for any metric \( g \) in \( \mathcal{M}_\Omega \) and any \( f \) in \( T_g \mathcal{M}_\Omega \), where \( L = \delta \delta D^{-1} d \) denotes the fourth order Lichnerowicz operator relative to \( g \). In particular, \( \sigma \) is closed.

**Proof.** From (4.4.1) and (3.2.6), we readily infer

\[
D_f \hat{s} = \hat{s}_f - (df, ds_g) = -2L(f),
\]

for any \( g \) in \( \mathcal{M}_\Omega \) and any \( f \) in \( T_g \mathcal{M}_\Omega \), which is (4.10.4) (recall that \( \bar{s} \) is constant on \( \mathcal{M}_\Omega \)). From (4.10.2), we then obtain

\[
(D_f \sigma)(f) = \langle D_f \hat{s}, f \rangle_g = -2(L(f_1), f_2)_g,
\]

for any \( f_1, f_2 \) in \( \mathcal{M}_\Omega \), hence (4.10.5). Since \( L \) is self-adjoint, we infer (\( D_f \sigma f_2 = (D_f \sigma) f_1 \), meaning that \( \sigma \) is closed. □
Since $\mathcal{M}_\Omega$ is contractible, $\sigma$ is actually exact and we define the Mabuchi K-energy, $E$, by

$$d\sigma = -dE,$$

hence by

$$dE_{\omega_0}(f) = -\int_M f s_g v_g,$$

for any $(g, \omega)$ in $\mathcal{M}_\Omega$ and any $f$ in $T_g\mathcal{M}_\Omega$ (identified with the space of real smooth functions on $M$ of mean value zero with respect to $v_g$). The Mabuchi K-energy $E$ defined that way is only determined up to an additive constant.

For any chosen $\omega_0$ in $\mathcal{M}_\Omega$, we then define $E_{\omega_0}$ as the unique determination of $E$ which vanishes at $\omega_0$. We then have:

$$E_{\omega_0}(\omega) = -\int_0^1 \sigma(\dot{c}) dt,$$

for any $\omega$ in $\mathcal{M}_\Omega$ and any curve $c : [0, 1] \to \mathcal{M}_\Omega$ connecting $\omega_0 = c(0)$ to $\omega = c(1)$.

**Remark 4.10.1.** In (4.10.7), $f$ is assumed to be of mean value zero with respect to $v_g$. If, instead, $T_g\mathcal{M}_\Omega$ is identified with the quotient $C^\infty(M, \mathbb{R})/\mathbb{R}$, in (4.10.7), $s_g$ must be replaced by $s_g - \bar{s}$, where $\bar{s} = \frac{\int_M s_g}{\mathcal{M}_\Omega}$ is the mean value of $s_g$ (the same for all elements of $\mathcal{M}_\Omega$). In the sequel, $E_{\omega_0}$ will be occasionally called the K-energy centered at $\omega_0$.

By choosing the path $c$ between $\omega_0$ and $\omega = \omega_0 + td\phi$ equal to the linear interpolation $\omega_t = \omega_0 + t\phi$, we get the following expression for $E_{\omega_0}$:

$$E_{\omega_0}(\phi) = -\int_M \phi \left( \int_0^1 (s_t - \bar{s}) v_t dt \right).$$

Notice that we must here integrate $\phi$ along $\int_0^1 (s_t - \bar{s}) v_t dt$, and not along $\int_0^1 s_t v_t$, as the segment $t \mapsto t\phi$ in $\mathcal{M}_{\omega_0}$ is *never* contained in $\mathcal{M}_{\omega_0}(0)$, unless $\phi \equiv 0$ (cf., Remark 4.10.1 above). It readily follows from (4.10.5) that $E_\Omega$ is convex, meaning that its hessian with respect to $D$ is everywhere non-negative. More precisely, at every point $g$ of $\mathcal{M}_\Omega$ and every non-zero vector $f$ in $T_g\mathcal{M}_\Omega$, we have that

$$\langle D_f dE_{\omega_0}, f \rangle = 2 \langle D^- df, D^- df \rangle_g \geq 0,$$

with equality if and only if $f$ is the momentum of a hamiltonian Killing vector field, [142].

Moreover, since $dE_{\omega_0} = -\sigma$, the critical points of $E_{\omega_0}$ are exactly the metric of constant scalar curvature in $\mathcal{M}_\Omega$.

**Remark 4.10.2.** Since $\bar{s}$ is the riemannian dual of $\sigma$, we also have

$$D_f \bar{s} = -2\delta \delta D^- df,$$

for each $f$ in $C^\infty(M, \mathbb{R}) = T_\phi \mathcal{M}_{\omega_0}$. From (4.10.3), we then recover

$$dC_\phi = -4\delta \delta D^- ds_g v_g.$$
4.11. The Futaki-Mabuchi bilinear form

Let \((M, g, J, \omega)\) be any (connected) compact Kähler manifold. Recall — cf. Chapter 2 — that any element \(X\) of the Lie algebra \(\mathfrak{h}\) of (real) holomorphic vector fields on \(M\) can be written as \(X = X_H + \text{grad} f^X_g + J\text{grad} h^X_g\), where \(X_H\) is the dual of a \(g\)-harmonic 1-form and where \(f^X_g, h^X_g\) are real functions normalized by \(\int_M f^X_g v_g = \int_M h^X_g v_g = 0\), and that the complex function \(F^X_g := f^X_g + ih^X_g\) has been called the complex potential of \(X\) with respect to \(g\) (we then have \(\bar{F}^X_g = iF^X_g\)). Also recall that the subspace of \(\mathfrak{h}\) defined by the condition \(X_H = 0\) is actually a Lie subalgebra of \(\mathfrak{h}\) denoted by \(\mathfrak{h}_{\text{red}}\), independent of \(g\), cf. Section 2.4.

**Proposition 4.11.1 (A. Futaki-T. Mabuchi [88]).** Let \(X_1, \ldots, X_r\) be any \(r\) elements of \(\mathfrak{h}_{\text{red}}\) and denote by \(F^X_{g_1}, \ldots, F^X_{g_r}\) the corresponding complex potentials with respect to some metric \(g\) of \(\mathcal{M}_\Omega\), normalized by \(\int_M F^X_{g_k} v_g = 0\), \(k = 1, \ldots, r\). For each \(k = 1, \ldots, r\), let \(\Phi_k\) be any holomorphic function defined on some open subset of \(\mathbb{C}\) containing the range of \(F^X_{g_k}\), so that \(\Phi_k(F^X_{g_k})\) is a well-defined complex function on \(M\). Then

\[
\Phi(X_1, \Phi_1, \ldots, X_r, \Phi_r) := \int_M \Phi_1(F^X_{g_1}) \cdots \Phi_r(F^X_{g_r}) v_g
\]

is independent of the choice of \(g\) in \(\mathcal{M}_\Omega\).

**Proof.** Let \(\omega_t = \omega_0 + d\bar{d}^c \phi_t\) be any curve in \(\mathcal{M}_\Omega\). By Lemma 4.5.1 and Lemma 4.5.1, the corresponding variation of \(\Phi_t := \Phi(X_1, \Phi_1, \ldots, X_r, \Phi_r)(g_t)\) is given by

\[
\frac{d \Phi_t}{dt} = \int_M \sum_{k=1}^r \Phi_1(F^X_{g_t}) \cdots \Phi'_k(F^X_{g_t}) \cdots \Phi_r(F^X_{g_t}) \bar{F}^X_{g_t} v_{g_t}
\]

\[- \int_M \Phi_1(F^X_{g_t}) \cdots \Phi_r(F^X_{g_t}) \Delta_{g_t} \phi_t v_{g_t}
\]

\[= 2 \int_M \sum_{k=1}^r \Phi_1(F^X_{g_t}) \cdots \Phi'_k(F^X_{g_t}) \cdots \Phi_r(F^X_{g_t}) \mathcal{L}_{X_k} \phi_t v_{g_t}
\]

\[- \int_M \Phi_1(F^X_{g_t}) \cdots \Phi_r(F^X_{g_t}) \Delta_{g_t} \phi_t v_{g_t}
\]

\[= 2(\sum_{k=1}^r \Phi_1(F^X_{g_t}) \cdots \Phi'_k(F^X_{g_t}) \cdots \Phi_r(F^X_{g_t}) \bar{\partial} F^X_{g_t}, \bar{\partial} \phi_t)
\]

\[- \langle \Phi_1(F^X_{g_t}) \cdots \Phi_r(F^X_{g_t}), \Delta_{g_t} \phi_t \rangle
\]

\[= 2\langle \bar{\partial} (\Phi_1(F^X_{g_t}) \cdots \Phi_r(F^X_{g_t})), \bar{\partial} \phi_t \rangle - \langle \Phi_1(F^X_{g_t}) \cdots \Phi_r(F^X_{g_t}), \Delta_{g_t} \phi_t \rangle
\]

\[= \langle \Phi_1(F^X_{g_t}) \cdots \Phi_r(F^X_{g_t}), (\bar{\partial} - \Delta) \phi_t \rangle = 0
\]

by Proposition 1.15.1 (here, \(\langle \cdot, \cdot \rangle\) denotes the (global) hermitian inner product and we have used the fact that the hermitian dual of \(X^k_{1.0}\) is equal to \(\bar{\partial} F^X_{g_t}\) — cf. Remark 2.5.1 — so that \(\mathcal{L}_{X_{1.0}^k} \phi_t = (\bar{\partial} F^X_{g_t}, \bar{\partial} \phi_t)\), where, again, \(\langle \cdot, \cdot \rangle\) denotes the (pointwise) hermitian inner product, cf. Section 1.3). \(\square\)
DEFINITION 6. The Futaki-Mabuchi bilinear form relative to $\Omega$ is the $\mathbb{C}$-bilinear form $B_\Omega$ defined on $\mathfrak{h}_{\text{red}}$ by

$$B_\Omega(X_1, X_2) = \int_M F_1 F_2 v_g,$$

for any two elements $X_1, X_2$ of $\mathfrak{h}_{\text{red}}$, of complex potentials $F_1$ and $F_2$ respectively with respect to some metric $g$ in $\mathcal{M}_\Omega$ (normalized by (3.1.3)).

By Proposition 4.11.1, this expression only depends on $X_1, X_2$ and $\Omega$, not upon the choice of $g$ in $\mathcal{M}_\Omega$, cf. [87].

The real part of $B_\Omega$ will be denoted by $B_\Omega$: we thus have

$$B_\Omega(X_1, X_2) = \int_M \Re(F_1 F_2) v_g$$

$$= \langle f_1, f_2 \rangle - \langle h_1, h_2 \rangle.$$  

Notice that

$$B_\Omega(JX_1, JX_2) = -B_\Omega(X_1, X_2).$$

For any choice of $g$ in $\mathcal{M}_\Omega$, $\mathfrak{k}_{\text{ham}}$ and $\mathfrak{j}_0$ are $B_\Omega$-orthogonal, the restriction of $B_\Omega$ to $\mathfrak{k}_{\text{ham}}$ is negative definite and its restriction to $\mathfrak{j}_0$ is positive definite; in particular, the restriction of $B_\Omega$ to the sum $\mathfrak{k}_{\text{ham}} \oplus \mathfrak{j}_0$ is non-degenerate. Notice that $B_\Omega$ is $H(M)$ invariant:

$$B_\Omega(\gamma^* X_1, \gamma^* X_2) = B_\Omega(X_1, X_2),$$

for any $\gamma$ in $H(M)$; this is because, if $\xi = \xi_0 + df + d^c h$ is the Hodge decomposition of the dual of $X$ with respect to some $g$ in $\mathcal{M}_\Omega$, then the Hodge decomposition of $\gamma^* \xi$ with respect to $\gamma^* g$ — which still belongs to $\mathcal{M}_\Omega$ — is $\gamma^* \xi = \gamma^* \xi_0 + d(f \circ \gamma) + d^c (h \circ \gamma)$. In particular, we have that:

$$B_\Omega([X_1, X_3], X_2) + B_\Omega(X_1, [X_2, X_3]) = 0,$$

for any $X_1, X_2$ in $\mathfrak{h}_0$ and any $X_3$ in $\mathfrak{h}$.

For more information concerning $B_\Omega$ in particular its link with the Killing form of $\mathfrak{h}_0$, see [87].

REMARK 4.11.1. For any $X_1, X_2$ in $\mathfrak{h}_{\text{red}}$, $B_\Omega(X_1, X_2)$ can be rewritten as

$$B_\Omega(X_1, X_2) = \langle \hat{X}_1, \hat{X}_2 \rangle_g - \langle JX_1, JX_2 \rangle_g,$$

for any $g$ in $\mathcal{M}_\Omega$, where, we recall, $\hat{X}_1, \hat{X}_2$ etc. denote the induced vector fields on $\mathcal{M}_\Omega$ — given by (3.1.6) — and $\langle \hat{X}_1, \hat{X}_2 \rangle_g$ etc. denotes the Mabuchi inner product at $g$ defined by (4.3.1). The fact that the rhs of (4.11.7) is actually independent of $g$ — which has been previously deduced from Proposition 4.11.1 — is then a straightforward consequence of the expression (4.5.6) of the covariant derivative of $\hat{X}_1$ etc. Indeed, for any $X$ in $\mathfrak{h}_{\text{red}}$, (4.5.6) can be simply read as

$$D_f \hat{X} = -\delta^g(f^c h^X_g),$$

whereas the derivative of the rhs of (4.11.7) at $g$ along any $f$ in $T_g \mathcal{M}_\Omega$ is equal to $\langle D_f \hat{X}_1, \hat{X}_2 \rangle + \langle \hat{X}_1, D_f \hat{X}_2 \rangle - \langle D_f JX_1, JX_2 \rangle - \langle JX_1, D_f JX_2 \rangle$; this can then be rewritten as

$$-\langle f, (d^c h^X_g, df^X_g) + (dh^X_g, d^c f^X_g) \rangle + (d^c f^X_g, dh^X_g) + (df^X_g, dh^X_g).$$
which is evidently equal to zero.

### 4.12. The Futaki character

Recall that for any fixed $X$ in $\mathfrak{h}$, the assignment $g \mapsto f^X_g$, where $f^X_g$ may be considered as a vector field on the Fréchet manifold $\mathcal{M}_\Omega$, and that this vector field is equal to $-\dot{X}$, where $\dot{X}$ denotes the vector field corresponding to the infinitesimal action of $H(M,J)$ on $\mathcal{M}_\Omega$, see Section 3.1. For any such fixed vector field $X$ in $\mathfrak{h}$, we then define $F_X : \mathcal{M}_\Omega \rightarrow \mathbb{R}$ by

$$F_X(g) = \int_M f^X_g s_g v_g,$$

where $s_g$ denotes the scalar curvature, $v_g$ the volume form of $g$.

By considering the 1-form $\sigma$ on $\mathcal{M}_\Omega$ introduced in Section 4.10, this can be rewritten as

$$F_X(g) = \sigma_g(\dot{X}).$$

**Proposition 4.12.1.** For any $X$ in $\mathfrak{h}$, $F_X$ is constant on $\mathcal{M}_\Omega$.

**Proof.** We denote by $a$ a dot the directional derivative at $g$ along $\dot{\phi}$ of $F_X$ and of all quantities appearing in the expression of $F_X$; $\dot{s}_g$ and $\dot{v}_g$ are given by lemma 3.2.1, whereas, by lemma 4.5.1, $\dddot{f}_g = \mathcal{L}_X \dot{\phi}$; we thus obtain

$$F_X = \int_M (\dddot{f}_g s_g + f^X_g \dddot{s} - f^X_g s_g \Delta \dot{\phi}) v_g$$

$$= \int_M (s_g \mathcal{L}_X \dot{\phi} - 2f^X_g \delta d - d \delta \dot{\phi} + f^X_g (d \dot{\phi}, ds_g) - f^X_g s_g \Delta \dot{\phi}) v_g,$$

where we have used (1.23.14). By integrating by parts, we get: $\int_M (f^X_g (d \dot{\phi}, ds_g) - f^X_g s_g \Delta \dot{\phi}) v_g = -\int_M s_g (df^X_g, \dot{\phi}) v_g$; we thus obtain:

$$\dddot{F}_X = \int_M (-2f^X_g \delta d - d \delta \dot{\phi} + s_g (d \dot{\phi}, \xi - df^X_g)) v_g,$$

or else, since $\xi - df = \xi_H + d^\ast h$ is co-closed and $L = \delta d$ is self-adjoint:

$$\dddot{F}_X = -2(\dot{\phi}, L(f^X_g) + \frac{1}{2}(ds_g, \xi - df^X_g)) = 0,$$

by (2.6.4).

**Remark 4.12.1.** An alternative proof of Proposition 4.12.1, due to J.-P. Bourguignon [39] — cf. also [22] — directly relies on the geometry of $\mathcal{M}_\Omega$ and the expression (4.12.2) of $F_X(g)$ in terms of the 1-form $\sigma$ described in Section 4.10. Since $\sigma$ is $H(M,J)$-invariant, we have that $\mathcal{L}_X \sigma = 0$, for each $X$ in $\mathfrak{h}$. By Cartan formula, $\mathcal{L}_X \sigma = \dot{X} \cdot d \sigma + d(\sigma(\dot{X}))$. Since $\sigma$ is closed we get $d(\sigma(\dot{X})) = 0$, which is exactly what Proposition 4.12.1 says.

In view of Proposition 4.12.1, the Futaki functional will be simply denoted by $F_\Omega$; we then have: $F_\Omega(X) := F_X(g)$, for any $X$ in $\mathfrak{h}$ and any $g$ in $\mathcal{M}_\Omega$. We will sometimes prefer the normalized Futaki functional, $\tilde{F}_\Omega$, defined by

$$\tilde{F}_\Omega = \frac{F_\Omega}{V_\Omega},$$
where, we recall, \( V_\Omega \) denotes the total volume of \( M \) with respect to any metric in \( \mathcal{M}_\Omega \).

The Futaki functional is \( \text{H}(M,J) \)-invariant:

\[
\mathcal{F}_\Omega(\gamma^*X) = \mathcal{F}_\Omega(X),
\]
for any \( \gamma \) in \( \text{H}(M,J) \) and any \( X \) in \( \mathfrak{h} \) (same argument as for \( B_\Omega \)). It follows that \( \mathcal{F}_\Omega \) is a character of the Lie algebra \( \mathfrak{h} \), i.e. satisfies

\[
\mathcal{F}_\Omega([X_1,X_2]) = 0,
\]
for any pair \( X_1, X_2 \) in \( \mathfrak{h} \). Accordingly, \( \mathcal{F}_\Omega \) is called the Futaki character of \((M,J)\).

One of the main application of the Futaki character has been to provide an obstruction to the existence of metric of constant scalar curvature within a given \( \text{Kähler} \) class. More precisely:

**Theorem 4.12.1** (A. Futaki [86], E. Calabi [46]). If \( \mathcal{M}_\Omega \) contains a metric \( g \) with constant scalar curvature, in particular a \( \text{Kähler-Einstein} \) metric, then \( \mathcal{F}_\Omega = 0 \).

Conversely, if \( \mathcal{F}_\Omega = 0 \), any extremal metric in \( \mathcal{M}_\Omega \) — if any — is of constant scalar curvature.

If \( \Omega \) is a multiple of the first Chern class of \((M,J)\) and \( \mathcal{F}_\Omega = 0 \), then any extremal metric in \( \mathcal{M}_\Omega \) — if any — is \( \text{Kähler-Einstein} \).

\[\text{Proof.}\] The first assertion readily follows from the definition (4.12.1) of \( \mathcal{F}_\Omega \) (recall that the real potential \( f_g^X \) is normalized by \( \int_M f_g^X v_g = 0 \)). The second assertion is also an easy consequence of (4.12.1). Indeed, if \( g \) is extremal, its scalar curvature \( s_g \) is a real potential for the (real) holomorphic vector field \(-JK = \text{grad}_{\bar{g}} s_g\), as well as \( s_g - \bar{s} \), which integrates to 0. If \( \mathcal{F}_\Omega \equiv 0 \), we have

\[0 = \mathcal{F}_\Omega(-JK) = \mathcal{F}_{-JK}(g) = \int_M (s_g - \bar{s}) s_g v_g = \int_M (s_g - \bar{s})^2 v_g.\]

It follows that \( s_g = \bar{s} \), i.e. that \( s_g \) is constant. If, moreover, \( \Omega = k c_1(M,J) \), for some (non-zero) real number \( k \), then the Ricci form \( \rho \) is harmonic, as \( s_g \) is constant — cf. Section 1.19 — and de Rham cohomologous to \( k \omega \): this yields \( \rho = k \omega \). \( \square \)

**Remark 4.12.2.** The third assertion in the above theorem has been first observed by S. Kobayashi in [118].

**Remark 4.12.3.** It readily follows from Theorem 4.12.1 that if \( \mathcal{M}_\Omega \) admits two extremal metrics, then both are of constant scalar curvature or both are of non-constant scalar curvature, according as \( \mathcal{F}_\Omega = 0 \) or \( \mathcal{F}_\Omega \neq 0 \).

**Remark 4.12.4.** It is a direct consequence of (4.12.2) that \( \mathcal{F}_\Omega = 0 \) if and only if \( \sigma(\hat{X}) = 0 \) for each \( X \) in \( \mathfrak{h} \). Since \( \sigma \) is \( \text{H}(M,J_0) \)-invariant, the latter condition just means that \( \sigma \) is basic relatively to the action of \( \text{H}(M,J_0) \), i.e. is the pull-back of a 1-form defined on the quotient \( \mathcal{M}_\Omega/\text{H}(M,J_0) \). Equivalently, the Futaki character \( \mathcal{F}_\Omega \) is zero if and only if the Mabuchi functional is constant on each orbit of \( \text{H}(M,J_0) \).

The Futaki character admits the following alternative formulation, due to E. Calabi [46]:

\[
\sigma(\hat{X}) = 0, \quad \forall X \in \mathfrak{h}.
\]
Proposition 4.12.2. The Futaki character can be alternatively defined by:

\[ \mathcal{F}_\Omega(X) = \int_M (X \cdot p_g) v_g, \]

for any \( X \) in \( \mathfrak{h} \), where \( p_g \) denotes the Ricci potential defined in Section 1.19.

Proof. This follows from the following simple computation:

\[
\int_M (X \cdot p_g) v_g = \int_M (dp_g, \xi) v_g = \int_M (dp_g, df) v_g = \int_M f \Delta p_g v_g = \int_M f (s_g - \bar{s}) v_g = \int_M f s_g v_g = \mathcal{F}_\Omega(X).
\]

Remark 4.12.5. The space \( \mathfrak{h}^* \) can be equivalently interpreted as the \( \mathbb{R} \)-dual of \( \mathfrak{h} \), viewed as a real vector space — this is the viewpoint developed here — or as the \( \mathbb{C} \)-dual of \( \mathfrak{h} \), viewed as a complex vector space (this amounts to identifying a complex linear form to its real part, i.e. to identifying \( \alpha \), a \( \mathbb{R} \)-linear map from \( \mathfrak{h} \) to \( \mathbb{R} \), to \( \frac{1}{2}(\alpha - i \alpha \circ J) \) and vice versa). In the latter acceptation the (complex) Futaki character is defined by

\[ \mathcal{F}_\Omega(X) = \int_M s_g F^X_g v_g = \int_M (X^{1,0} \cdot p_g) v_g, \]

where \( F^X_g = f^X_g + ih^X_g \) is the complex potential of \( X \). In the sequel, unless otherwise specified, \( \mathcal{F}_\Omega \) will always denote the (real) Futaki character defined by (4.12.1). Note that, according to this convention, \( \mathcal{F}_\Omega(X) = 0 \), whenever \( X \) is Killing with respect to some metric \( g \) in \( \mathcal{M}_\Omega \), as \( f^X_g \) is then identically zero.

More information concerning the Futaki character can be found in [86] and references therein.

4.13. The extremal vector field of a Kähler class

In this section, we consider metrics in \( \mathcal{M}_\Omega \) which are invariant under some maximal connected compact subgroup of \( H_{\text{red}}(M, J) \), where, we recall, \( H_{\text{red}}(M, J) \) denotes the reduced automorphism group of \( (M, J) \) — see Section 2.4 — and we denote by \( \mathcal{M}_{\Omega}^{\text{max}} \) the space of these metrics.

For any chosen maximal connected compact subgroup, say \( G \), of \( H_{\text{red}}(M, J) \), we denote by \( \mathcal{M}_{\Omega}^G \) the space of \( G \)-invariant elements of \( \mathcal{M}_{\Omega} \). Then

\[ \mathcal{M}_{\Omega}^{\text{max}} = \bigcup_{h \in H_{\text{red}}(M, J)/G} \mathcal{M}_{\Omega}^{hGh^{-1}}. \]

Notice that by Theorem 3.5.1 any extremal metric — if any — belongs to \( \mathcal{M}_{\Omega}^{\text{max}} \), hence to the \( H_{\text{red}}(M, J) \)-orbit of an extremal metric in \( \mathcal{M}_{\Omega}^G \).

We henceforth restrict our attention to \( \mathcal{M}_{\Omega}^G \), for some chosen maximal connected compact subgroup \( G \) of \( H_{\text{red}}(M, J) \). We denote by \( \mathfrak{g} \subset \mathfrak{h}_{\text{red}} \) the Lie algebra of \( G \); \( \mathfrak{g} \) is then the space of hamiltonian Killing vector fields for all metrics \( g \) in \( \mathcal{M}_{\Omega}^G \).

The space \( \mathcal{M}_{\Omega}^G \) is connected and \( \mathcal{D} \)-totally geodesic in \( \mathcal{M}_{\Omega} \) as the set of fixed points by an isometry group.
For any chosen metric \( g \) in \( \mathcal{M}_\Omega^G \) we denote by \( P_g^G \) the space of Killing potentials relative to \( g \), i.e., the kernel of the fourth order Lichnerowicz operator \( L = \delta\delta D^{-1} d \). cf. Section 2.2: \( P_g^G \) is then the direct sum of \( \mathbb{R} \), the space of constant real functions on \( M \), and of the space of real potentials of elements of \( J_g \subset h_{\text{red}} \). Moreover, \( P_g^G \) is stable for the Poisson bracket determined by the Kähler form \( \omega \) of \( g \) and, as a Lie algebra, is isomorphic to Lie algebra direct sum \( \mathbb{R} \oplus g \).

We denote by \( \Pi_g^G \) the orthogonal projector with respect to the global inner product \( \langle \cdot, \cdot \rangle \) of the (real) Hilbert \( L^2(M, \mathbb{R}) \) of real \( L^2 \)-functions on \( M \) onto \( P_g^G \) (recall that \( L^2(M, \mathbb{R}) \) is defined as the completion of the space \( C^\infty(M, \mathbb{R}) \) of smooth real functions on \( M \) for the inner product \( \langle \cdot, \cdot \rangle \)).

Notice that \( P_g^G \) is isomorphic to \( \mathbb{R} \oplus g \) for any \( g \) in \( \mathcal{M}_\Omega^G \) but, as a subspace of \( C^\infty(M, \mathbb{R}) \), depends on \( g \), as does \( \Pi_g^G \).

For any real function \( f \), \( \Pi_g^G(f) \) will be called the Killing part of \( f \) with respect to \( g \). In particular, for any metric \( g \) in \( \mathcal{M}_\Omega \), \( \Pi_g^G(s_g) \) is called the Killing part of the scalar curvature of \( g \) and we thus get we get the following \( L^2 \)-orthogonal decomposition of \( s_g \):

\[
 s_g = s_g^G + \Pi_g^G(s_g),
\]

where \( s_g^G \) — call it the reduced scalar curvature with respect to \( G \) — is \( L^2 \)-orthogonal to \( P_g^G \). Observe that \( g \) is extremal if and only if the reduced scalar curvature \( s_g^G \) is identically zero.

We now define a vector field \( Z^G_\Omega \) in \( J_g \) by

\[
 Z^G_\Omega(X) = B_\Omega(X, Z^G_\Omega),
\]

for any \( X \) in \( J_g \), where, we recall, \( F_\Omega \) and \( B_\Omega \) denote the Futaki character and the Futaki-Mabuchi bilinear form of \( \Omega \).

Notice that \( Z^G_\Omega \) is well-defined, as the restriction of \( B_\Omega \) to \( J_g \) is positive definite. Moreover, \( Z^G_\Omega \) and is \( G \)-invariant, so that \( JZ^G_\Omega \) belongs to the center of \( g \).

By its very definition, \( Z^G_\Omega \) only depends on the Kähler class \( \Omega \) and of the choice of \( G \). Following \[87\], we call it the extremal vector field of \( \Omega \) relative to \( G \).

Any other maximal compact subgroup, \( G' \), of \( H_{\text{red}}(M, J) \) is conjugate to \( G \) by an element, \( \gamma \) say, of \( H_{\text{red}}(M, J) \); since \( F_\Omega \) and \( B_\Omega \) are both \( H(M, J) \)-invariant, we infer:

\[
 Z^G_\Omega = \gamma \cdot Z^G_\Omega.
\]

We then have the following proposition, cf. \[107\]:

**Proposition 4.13.1.** For any metric \( g \) in \( \mathcal{M}_\Omega^G \), the extremal vector field \( Z^G_\Omega \) is the gradient of the Killing part of the scalar curvature:

\[
 Z^G_\Omega = \nabla_{g}(\Pi_g^G(s_g)).
\]

Moreover, the \( L^2 \)-square norm of \( \Pi_g^G(s_g) \) is given by

\[
 \int_M (\Pi_g^G(s_g))^2 v_g = E_\Omega,
\]
Remark 4.13.1. All extremal metrics in $\mathcal{M}_\Omega$ belong to $\mathcal{M}^\text{max}_\Omega$ by Theorem 3.5.1. It then follows from Proposition 4.13.1 that

$$C(g) = \mathcal{E}(\Omega)$$

(4.13.11)

for any extremal metric $g$ in $\mathcal{M}_\Omega$. It has been recently shown by X. X. Chen that $\mathcal{E}(\Omega)$ is a lower bound for $C$ on the whole space $\mathcal{M}_\Omega$ [56]. The same lower bound for $C$ on $\mathcal{M}_\Omega$ have been obtained by S. Donaldson, together with an improved lower bounds in the case when $\Omega$ admits no extremal metrics, in the polarized case, i.e. when $\Omega$ is an integral class, hence the Chern class of an ample holomorphic line bundle [79].

Remark 4.13.2. For any (connected) compact subgroup $G$ of $H_{\text{red}}(M, J)$ — not necessarily maximal — the space, $\mathcal{M}^G_\Omega$, of $G$-invariant metrics in $\mathcal{M}_\Omega$ is still a totally geodesic submanifold of $\mathcal{M}_\Omega$ and the Lie algebra $\mathfrak{g}$ of $G$ is still a Lie algebra of hamiltonian Killing vector fields for all metrics $g$ in $\mathcal{M}^G_\Omega$, although some of them may now have more hamiltonian Killing vector fields than the ones in $\mathfrak{g}$. For any metric $g$ in $\mathcal{M}^G_\Omega$, denote by $P^G_\mathfrak{g}$ the space

(4.13.7) \[ \mathcal{E}(\Omega) = \frac{S^G_\Omega}{V_\Omega} + \mathcal{F}_\Omega(\Omega^G_\Omega) \]

In particular, $\int_M (\Pi^G_g(s_g))^2 v_g$ is independent of $g$ in $\mathcal{M}^G_\Omega$ and of $G$ in $H_{\text{red}}(M, J)$ and provides the following lower bound for the restriction of the Calabi functional $C$ to $\mathcal{M}^\text{max}_\Omega$.

For any (connected) compact subgroup $G$ of $H_{\text{red}}(M, J)$, by setting

$$B_{\Omega}(\Omega^G_\Omega, X) = \int_M f^G_g f^X_g v_g = \mathcal{F}_\Omega(X) = \int_M s_g f^X_g v_g,$$

for any $X$ in $J_g$, of real potential $f^X_g$ (of mean value 0). It follows that $f^G_g = \Pi^G_g(s_g)$ up to an additive constant. We thus get (4.13.5). Notice that the potential of $\Omega^G_\Omega$, as defined in Section 2.1, is $\Pi^G_g(s_g) - \bar{s}$; it follows that

$$\mathcal{F}_\Omega(\Omega^G_\Omega) = \int_M s_g (\Pi^G_g(s_g) - \bar{s}) v_g = \int_M |\Pi^G_g(s_g)|^2 v_g - \frac{S^G_\Omega}{V_\Omega}.$$

This gives (4.13.6), where $\mathcal{E}(\Omega)$ is defined by (4.13.7). By using (4.13.4) and (4.12.5), we infer that $\mathcal{E}_\Omega$ is actually independent of $G$ as well. From (4.13.2), we derive:

$$C(g) = \int_M (s^G_g)^2 v_g + \int_M |\Pi^G_g(s_g)|^2 v_g \geq \int_M |\Pi^G_g(s_g)|^2 v_g = \mathcal{E}(\Omega)$$

(4.13.10)

for any $g$ in $\mathcal{M}^G_\Omega$, with equality if and only if $s^G_g \equiv 0$, hence if and only if $g$ is extremal. Since any $g$ in $\mathcal{M}^\text{max}_\Omega$ belongs to $\mathcal{M}^G_\Omega$ for some maximal compact subgroup of $H_{\text{red}}(M, J)$, the last statement of Proposition 4.13.1 follows.

Proof. By its very definition, for any $g$ in $\mathcal{M}^G_\Omega$, $\Omega^G_\Omega = \text{grad}_g f^G_g$ for some real function $f^G_g$, determined, up to an additive constant, by

$$B_{\Omega}(\Omega^G_\Omega, X) = \int_M f^G_g f^X_g v_g = \mathcal{F}_\Omega(X) = \int_M s_g f^X_g v_g,$$
of Killing potentials, including constants, of elements of \( g \) with respect to \( g \)— \( \mathcal{L}_g \) is still a Lie algebra for the Poisson bracket, isomorphic to \( \mathbb{R} \oplus g \)— and by \( \Pi^G_g \) the corresponding orthogonal projector. In particular, \( \Pi^G_g(s_g) \) is called the Killing part of the scalar curvature \( s_g \) with respect to \( G \) and we still have the orthogonal decomposition (4.13.2) of \( s_g \) as the sum of its Killing part \( \Pi^G_g(s_g) \) and its reduced part \( s^G_g \).

We then defined an extremal vector field relative to \( G \), denoted by \( Z^G_G \Omega \), either by (4.13.3), or by \( Z^G_G \Omega = \text{grad}_g \Pi^G_g(s_g) \) for any \( g \) in \( \mathcal{M}^G_G \). The equivalence of the two definitions, in particular the fact that \( \text{grad}_g \Pi^G_g(s_g) \) is independent of \( g \) in \( \mathcal{M}^G_G \), is shown as in the proof of the first part of Proposition 4.13.1.

Moreover, \( s_g \) is \( G \)-invariant, as well as \( \Pi^G_g(s_g) \), so that \( JZ^G_G \Omega \) belongs to the center of \( g \).

If \( G = \{1\} \), the corresponding extremal vector field is identically zero, whereas, if \( G \) is a maximal compact subgroup of \( \text{H}_{\text{red}}(M,J) \), we simply get the extremal vector field previously defined in this section. It may be however of some interest to consider extremal vector field attached to non-maximal compact subgroups of \( \text{H}_{\text{red}}(M,J) \), in particular in the toric case — cf. e.g. Donaldson’s paper [77] — due in particular to the following simple observation:

**Proposition 4.13.2.** Let \( G \) be any maximal compact subgroup of \( \text{H}_{\text{red}}(M,J) \) and let \( T \) be any maximal torus in \( G \). Then,

\[
Z^G_T = Z^G_G.
\]

**Proof.** Let \( g \) be any metric in \( \mathcal{M}^G_G \). Then, \( g \) belongs to \( \mathcal{M}^T_T \) as well and, as already observed, \( JZ^G_G = J\text{grad}_g \Pi^G_g(s_g) \) belongs to the center of \( g \), hence to \( t = \text{Lie}(T) \). It follows that \( \Pi^G_g(s_g) \) belongs to \( \mathcal{P}^T \), hence is equal to \( \Pi^T_g(s_g) \). We thus have:

\[
Z^G_T = \text{grad}_g \Pi^G_g(s_g) = \text{grad}_g \Pi^T_g(s_g) = Z^T_T.
\]

Notice that in Proposition 4.13.2, \( T \) could have been replaced by any (connected) closed subgroup of \( G \) containing the identity component of the center of \( G \).

4.14. Relative Mabuchi K-energy and relative Futaki character

In this section we fix a (connected) compact subgroup, \( G \), of the reduced automorphism group \( \text{H}_{\text{red}}(M,J) \) and we define a K-energy relative to \( G \), denoted \( E^G_G \), for a chosen Kähler class \( \Omega \). When \( G = \{1\} \), this is simply the Mabuchi K-energy defined in Section 4.10, whereas in the case when \( G \) is a maximal compact subgroup of \( \text{H}_{\text{red}}(M,J) \), \( E^G \) is the relative, or modified, K-energy introduced independently by D. Guan in [96] and by S. Simanca in [175], which plays the same role for the search of a general extremal metric within \( \Omega \) as the Mabuchi K-energy for Kähler metrics of constant scalar curvature, cf. . We similarly define a Futaki character relative to, \( G \), whose vanishing is a necessary condition that \( \Omega \) can possibly admit an extremal metric if \( G \) is a maximal compact subgroup \( G_{\text{max}} \) of \( \text{H}_{\text{red}}(M,J) \) or a maximal torus of \( G_{\text{max}} \).
4.14. RELATIVE MABUCHI K-ENERGY AND RELATIVE FUTAKI CHARACTER

We already observed — cf. Remark 4.13.2 — that an extremal vector field \( Z_G \) can be defined for any (connected) compact subgroup \( G \) of \( H_{\text{red}}(M, J) \) and that \( JZ_G \) then belongs to the center of \( \mathfrak{g} \). The induced vector field \( \hat{Z}_G \) on \( M_\Omega \) is then tangent to \( M_{\Omega}^G \). Recall that \( \hat{Z}_G \) has the following expression:

\[
\hat{Z}_G(g) = \Pi_g^G(s_g),
\]

for any \( g \in M_{\Omega}^G \). We denote by \( \zeta_G \) the corresponding dual 1-form on \( M_{\Omega}^G \):

\[
\zeta_G(g) = \Pi_g^G(s_g)v_g,
\]

for any \( g \in M_{\Omega}^G \).

**Lemma 4.14.1.** The restriction of \( \hat{Z}_G \) to \( M_{\Omega}^G \) and its dual 1-form \( \zeta_G \) are \( \mathcal{D} \)-parallel. In particular, \( \zeta_G \) is closed.

**Proof.** \( JZ_G \) is a Hamiltonian Killing vector field for any \( g \in M_{\Omega}^G \); by Proposition 4.5.1, the restriction of \( \hat{Z}_G \) to \( M_{\Omega}^G \) is then \( \mathcal{D} \)-parallel. The dual 1-form \( \zeta_G \) is \( \mathcal{D} \)-parallel as well, hence closed. \( \square \)

We now define \( \tilde{\sigma} \) on \( M_{\Omega}^G \) by

\[
\tilde{\sigma}^G = \sigma|_{M_{\Omega}^G} - \zeta_G,
\]

where, we recall, \( \sigma \) denotes the 1-form on \( M_\Omega \) determined by the scalar curvature, namely the 1-form \( g \mapsto s_g v_g \).

Then, \( \sigma^G \) is a closed 1-form on \( M_{\Omega}^G \) and its covariant derivative coincides with the restriction to \( M_{\Omega}^G \) of the covariant derivative of \( \sigma \).

Moreover, we have that

\[
\sigma^G : g \mapsto s_g^G v_g,
\]

where, we recall, \( s_g^G \) denotes the reduced scalar curvature appearing in (4.13.2).

In particular, \( \sigma^G_g = 0 \) if and only if \( g \) is extremal.

The **relative Mabuchi K-energy**, \( E^G \), is defined as the opposite of the primitive of \( \tilde{\sigma} \):

\[
\tilde{\sigma} = -dE^G.
\]

The relative Mabuchi K-energy is defined up to an additive constant. For any chosen base-point \( \omega_0 \) in \( M_{\Omega}^G \), denote by \( E^G_{\omega_0} \) the relative K-energy which vanishes at \( \omega_0 \); we then have:

\[
E^G_{\omega_0}(\omega) = -\int_0^1 \tilde{\sigma}(\dot{c})dt,
\]

for any path \( c(t) \) from \( \omega_0 = c(0) \) to \( \omega = c(1) \). Since \( \tilde{\sigma} \) is closed and \( M_{\Omega}^G \) is contractible, \( E^G_{\omega_0}(\omega) \) is independent of the chosen path in \( M_{\Omega}^G \). In particular, by choosing \( c_t = t\phi \), corresponding to the curve \( \omega_t = \omega_0 + tdd^c\phi \) in \( M_\Omega \), we get

\[
E^G_{\omega_0}(\omega) = -\int_M \phi \left( \int_0^1 s_t^G v_t dt \right),
\]

where \( s_t \) and \( v_t \) are relative to the metric \( g_t \) determined by \( \omega_t \).
Proposition 4.14.1. At any point \( g \) of \( \mathcal{M}_\Omega \), the Hessian of the relative Mabuchi \( K \)-energy \( E^G \) is given by

\[
(4.14.8) \quad (\mathcal{D}_f dE^G)_g = 2\langle D^- df, D^- df \rangle_g
\]

for any \( G \)-invariant function \( f \), with \( \int_{\mathcal{M}} f \nu_g = 0 \), viewed as an element of \( T_g \mathcal{M}_\Omega \). In particular, \( \mathcal{D}dE^G_\Omega \) is everywhere non-negative and its kernel at \( g \) is the space of \( G \)-invariant momenta of hamiltonian \( g \)-Killing vector fields in \( \mathfrak{g} \). Moreover, the critical points of \( E^G_\Omega \) are exactly the \( G \)-invariant extremal metrics in \( \mathcal{M}_\Omega \).

Proof. Since \( \sigma^G \) differs from \( \sigma|_{\mathcal{M}^G_\Omega} \) by a \( \mathcal{D} \)-parallel 1-form, the hessian of \( E^G \) relatively to \( \mathcal{D} \) is the restriction to \( \mathcal{M}^G_\Omega \) of the hessian of \( E \) \((4.14.8)\) then is a direct consequence of \((4.10.10)\), by only considering \( G \)-invariant functions \( f \). The critical points of \( E \) are the metrics in \( \mathcal{M}^G_\Omega \) at which \( \hat{\sigma} \) vanishes, hence the \( G \)-invariant extremal metrics, as already observed. \( \square \)

For any (connected) compact subgroup \( G \) of \( H_{\text{red}}(M, J) \), we denote by \( H_G(M, J) \) the identity component of the normalizer of \( G \) in \( H_{\text{red}}(M, J) \) and by \( H^G(M, J) \) the identity component of its centralizer. We similarly denote by \( \mathfrak{h}_G \) the Lie algebra of \( H_G(M, J) \) and by \( \mathfrak{h}^G \) the Lie algebra of \( H^G(M, J) \). Hence \( H^G(M, J) \) is a Lie subgroup of \( H_G(M, J) \) and both act on the space \( \mathcal{M}^G_\Omega \) of \( G \)-invariant metrics in \( \mathcal{M}_\Omega \). More precisely, the action of \( H_G(M, J) \) on \( \mathcal{M}^G_\Omega \) factors through the quotient \( H_G(M, J)/G \), whereas the action of \( H^G(M, J) \) factors through the quotient \( H^G(M, J)/Z(G) \), where \( Z(G) \) denotes the intersection of \( H^G(M, J) \) with \( G \) (which is contained in the center of \( G \)). The two actions can then be considered as a same action, due to the following lemma:

Lemma 4.14.2. For any connected compact subgroup \( G \) of \( H_{\text{red}}(M, J) \) we have that

\[
(4.14.9) \quad H_G(M, J)/G = H^G(M, J)/Z(G).
\]

Proof. We first check that

\[
(4.14.10) \quad \mathfrak{h}_G = \mathfrak{h}^G + \mathfrak{g},
\]

where \( \mathfrak{g} \) denotes the Lie algebra of \( G \). Let \( Z = Z_H + \text{grad}_gf^Z + J\text{grad}_gh^Z \) be any element of \( \mathfrak{h}_G \) where \( g \) is any fixed metric in \( \mathcal{M}^G_\Omega \). Then, \( Z \) belongs to \( \mathfrak{h}_G \) if and only if \( [X, Z] = \mathcal{L}_X Z \) belongs to \( \mathfrak{g} \) for any \( X = J\text{grad}_gf^X \) in \( \mathfrak{g} \). Now, \( Z_H \) is the dual vector field of a harmonic 1-form, say \( \zeta_H \), so that \( \mathcal{L}_X \zeta_H = 0 \), cf. the proof of Propositions 2.2.1 and 2.3.2. Since, \( X \) is a Killing vector field, this implies that \( \mathcal{L}_X Z_H = 0 \), hence that \( Z_H \) is \( G \)-invariant. Since, \( X \) preserves \( G \) and \( J \), we eventually get: \( [X, Z] = \text{grad}_g(\mathcal{L}_X f^Z) + J\text{grad}_g(\mathcal{L}_X h^Z) \). This vector field clearly belongs to \( \mathfrak{g} \) for all \( X \) in \( \mathfrak{g} \) if and only if \( \mathcal{L}_X f^Z \) is constant, hence zero, for all \( X \) in \( \mathfrak{g} \), meaning that \( f^Z \) is \( G \)-invariant, and \( \mathcal{L}_X h^Z \) belongs to \( P^G_\mathfrak{g} \) for all \( X \) in \( \mathfrak{g} \). This implies that \( \gamma^*h^Z - h^Z \) belongs to \( P^G_\mathfrak{g} \) for any \( \gamma \) in \( G \). By integrating over \( G \) with respect to the bi-invariant measure of total volume 1 — this can be realized by using the volume-form \( \nu_G \) of any bi-invariant riemannian metric on \( G \), as \( G \) is a compact Lie group — we get that \( h^Z = \tilde{h}^Z + f \), where \( \tilde{h}^Z := \int_G \gamma^*h^Z \nu_G \) is \( G \)-invariant and \( f \) belongs to \( P^G_\mathfrak{g} \). We thus
end up with \( Z = (Z_H + \text{grad}_g^X f^Z + J\text{grad}_g^Z + J\text{grad}_g f) \), where \( \text{grad}_g f \) belongs to \( g \), whereas \( Z_H + \text{grad}_g^X f^Z + J\text{grad}_g^Z = Z - J\text{grad}_g f \) is \( G \)-invariant in \( h_G \), hence belongs to \( h^G \). This completes the proof of (4.14.10). From (4.14.10), we deduce that \( H_G(M, J)/G \) and its Lie subgroup \( H^G(M, J)/Z(G) \) have the same Lie algebra, hence coincide (a more precise formulation of this isomorphism will appears in Section 9.9). □

**Definition 7.** The Futaki character relative to \( G \), denoted by \( \mathcal{F}_G^\Omega \), is the \( \mathbb{R} \)-linear form on \( h_G/h = h^G/\mathfrak{h}(G) \) defined by

\[
\mathcal{F}_G^\Omega(X) = \int_M f_g^X s_g^G v_g,
\]

for any \( g \) in \( \mathcal{M}_G^\Omega \), where \( f_g^X \) denotes the real potential of \( X \) with respect to \( g \) and, we recall, \( s_g^G \) denotes the reduced scalar curvature of \( g \) with respect to \( G \).

This definition relies on the following proposition:

**Proposition 4.14.2.** The relative Futaki character \( \mathcal{F}_G^\Omega \) defined by (4.14.11) is independent of the choice of \( g \) in \( \mathcal{M}_G^\Omega \). It is equal to zero if and only if the relative K-energy \( E^G \) is constant on each orbit in \( \mathcal{M}_G^\Omega \) under the action of \( H_G(M, J)/G = H^G(M, J)/Z(G) \).

**Proof.** For any \( X \) in \( h^G \), denote by \( \hat{X} \) the induced vector field on \( \mathcal{M}_G^\Omega \). Then, \( \hat{X} \) is given by \( \hat{X}(g) = f_g^X \) for any \( g \) in \( \mathcal{M}_G^\Omega \), where \( f_g^X \) is \( G \)-invariant. Now, the closed 1-form \( \hat{s} \) is \( H_G(M, J) \)-invariant, so that \( \hat{s}(\hat{X}) = \int_M f_g^X s_g v_g \) is constant on \( \mathcal{M}_G^\Omega \) (cf. Remark 4.12.1). Moreover, since \( \hat{s}(\hat{X}) = -dE^G(\hat{X}) \), \( \mathcal{F}_G^\Omega \equiv 0 \) if and only \( E^G \) is constant on each \( H_G(M, J) \)-orbit in \( \mathcal{M}_G^\Omega \). □

In the case when \( G \) is a maximal compact subgroup of \( H_{\text{red}}(M, J) \) or a maximal torus, we get

**Proposition 4.14.3.** For any maximal compact subgroup \( G_{\text{max}} \) of \( H_0(M, J) \), the Futaki character \( \mathcal{F}_{G_{\text{max}}}^\Omega \) relative to \( G_{\text{max}} \) and the Futaki character \( \mathcal{F}_T^\Omega \) relative to any maximal torus \( T \) of \( G_{\text{max}} \) — cf. Remark 4.13.2 — is identically zero whenever there exists an extremal metric within the Kähler class \( \Omega \).

**Proof.** If there exists an extremal Kähler metric in \( \mathcal{M}_\Omega \), there exists one, \( g \), say, in \( \mathcal{M}_G^{\text{G}_{\text{max}}} \) by Calabi’s theorem 3.5.1. Then, \( s_g \) is a Killing potential, hence is equal to \( \Pi_g^\Omega(s_g) \), in fact in \( \Pi_g^T(s_g) \), cf. the proof of Proposition 4.13.2. It follows that the reduced scalar curvature \( s_g^{G_{\text{max}}} = s_g - \Pi_g^{G_{\text{max}}}(s_g) = s_g - \Pi_g^T(s_g) = s_g^T \) is identically zero. The rhs of (4.14.11) is then equal to 0 for any \( X \) in \( h_G \), for \( G = G_{\text{max}} \) and \( G = T \) and, more generally, for any subgroup \( G \) of \( G_{\text{max}} \) which contains the center of \( G_{\text{max}} \), cf. Remark 4.13.2. □
Remark 4.14.1. Since \( s^G_g = s_g - \Pi^G_g(s_g) \), the Futaki character relative to any (connected) compact subgroup \( G \) of \( H_{\text{red}}(M,J) \) is also given by

\[
F_G^\Omega(X) = F_\Omega(X) - B_\Omega(X, Z^G_\Omega) F_\Omega(Z^G_\Omega)
\]

(4.14.12)

(recall that \( F_\Omega(Z^G_\Omega) = B_\Omega(Z^G_\Omega, Z^G_\Omega) \)). The relative Futaki character as defined above then fits with the relative Futaki character introduced by G. Székelyhidi in [180].

4.15. Uniqueness of extremal Kähler metrics

The uniqueness issue for Kähler-Einstein metrics has been understood for a long time. The uniqueness, up to scaling, of Kähler-Einstein metrics with negative (constant) scalar curvature on a compact complex manifold of negative first Chern class is a part of Theorem 1.21.1, whereas uniqueness of Kähler-Einstein metrics with zero scalar curvature within any given Kähler class on a compact complex manifold of zero first Chern class directly follows from the Calabi-Yau theorem, cf. Section 1.20. In contrast with the negative and the zero case, no existence nor uniqueness theorem for Kähler-Einstein metrics of positive scalar curvature can be deduced from the Monge-Ampère equation (1.21.4). As a matter of fact, existence of Kähler metric of positive type on a Fano manifold is not granted in general — cf. Section 6.6 — whereas the uniqueness issue was solved by S. Bando and T. Mabuchi [23] in the following form:

Theorem 4.15.1. Let \((M, J)\) be a connected, compact manifold with positive first Chern class \( c_1(M,J) \). For any positive real number \( k \), consider the Kähler class \( \Omega = \frac{2\pi}{k} c_1(M,J) \) and assume that the space, \( \mathcal{E} \), of Kähler-Einstein metrics in \( M_\Omega \) is non-empty. Then,

(i) \( \mathcal{E} \) coincides with the orbit of any of its elements under the action of \( H(M) \).

(ii) For any \( \omega_0 \) in \( \mathcal{E} \) and any \( \omega \) in \( M_\Omega \), we have

\[
E_{\omega_0}(\omega) \geq 0,
\]

with equality if and only if \( \omega \) belongs to \( \mathcal{E} \).

Proof. The proof of this theorem goes beyond the scope of these notes; the reader is referred to the original paper [23], cf. also [86].

The beautiful argument used by S. Bando and T. Mabuchi in [23] heavily relies on the Kähler-Einstein assumption, hence does not generalize to Kähler metrics of constant scalar curvature, a fortiori to general extremal Kähler metrics. On the other hand, as first observed by S. Donaldson in [74], uniqueness issues for extremal Kähler metrics could be easily solved if the existence of geodesics between any two points in \( M_\Omega \) were granted, as evidenced by the following proposition:

Proposition 4.15.1 ([74] Proposition 10). Let \((M, J)\) be a compact complex manifold and fix \( \Omega \) a Kähler class on \( M \). Assume that \( M_\Omega \) contains a
Kähler metric of constant scalar curvature and denote by $E_{\omega_0}$ the Mabuchi K-energy which vanishes at $\omega_0$. Then:

(i) Any metric of constant scalar curvature in $\mathcal{M}_\Omega$ connected to $\omega_0$ by a geodesic belongs to the orbit of $\omega_0$ under the action of the reduced automorphism group $\mathbb{H}_{\text{red}}(M, J)$.

(ii) For any $\omega$ in $\mathcal{M}_\Omega$ connected to $\omega_0$ by a geodesic, we have:

$$E_{\omega_0}(\omega) \geq 0,$$

with equality if and only if $\omega$ is of constant scalar curvature.

**Proof.** Let $\omega$ be any element of $\mathcal{M}_\Omega$ liked to $\omega_0$ by a geodesic $\omega_t = \omega_0 + dt \phi_t$, with $I_{\omega_t}(\phi_t) = 0$ for all $t$ in the interval $[0, T]$. Let $T$ be the tangent vector field along this geodesic, so that $T_{\omega_t} = \phi_t$ in $C^\infty_{0, g_t}(M, \mathbb{R}) = T_{\omega_t}\mathcal{M}_\Omega$. From (4.10.5), we infer

$$\frac{d}{dt}\sigma_{\phi_t}(T) = (D_T\sigma)(T) + \sigma(D_T T)$$

(4.15.3)

$$= -2\int_M \phi_t \delta D^- d\phi_t v_t$$

(see [74], Proposition 10). Suppose now that $\omega = \omega_1$ is of constant scalar curvature. We then have: $\sigma_{\omega_0}(T) = \sigma_{\omega_1}(T) = 0$. Since the rhs of (4.15.3) is non-positive for each $t$, we conclude that $\sigma_{\omega_1}(T) = 0$ and $D^- d\phi_0 = 0$ for all $t$. This means that $X_t = -\text{grad}_{g_t}(\phi_t)$ is a (real) holomorphic vector field for each value of $t$ in $[0, 1]$. It then follows from Proposition 4.6.3 that $X_t = X_0$ for all $t$ and that $\omega = (\Phi^{X_0}_{t_0})^{*}\omega_0$, where $\Phi^{X_0}_{t_0}$ is the value at $t_1$ of the flow of $X_0$, in particular belongs to $\mathbb{H}_{\text{red}}(M, J)$. This completes the proof of (i). In order to prove (ii), we observe that the identity (4.15.3) can be rewritten as

$$\frac{d^2 E_{\omega_0}(\omega_t)}{dt^2} = 2\int_M |D^- d\phi_t|_{g_t}^2 v_t,$$

(4.15.4)

as $\sigma = -dE_{\omega_0}$. Now, $E_{\omega_0}(\omega_t) = 0$ by definition and, since $\omega_0$ is of constant scalar curvature, we also have $\frac{dE_{\omega_0}(\omega_1)}{dt}|_{t=0} = -\sigma_{\omega_0} = 0$, whereas, by (4.15.4), the second derivative of $E_{\omega_0}(\omega_t)$ is non-negative for all values of $t$ in $[0, 1]$; $E_{\omega_0}(\omega_t)$ is then non-negative along the geodesic, in particular $E_{\omega_0}(\omega) = E_{\omega_0}(\omega_1)$ is non-negative. Moreover, $E_{\omega_0}(\omega) = 0$ if and only if $E_{\omega_0}(\omega_t) = 0$ for each $t$; as we already saw, this happens if and only if $\omega$ belongs to the $\mathbb{H}_{\text{red}}(M, J)$-orbit of $\omega_0$, if and only if $\omega$ is of constant scalar curvature.

By considering the relative $K$-energy $E^G_{\omega_0}$, D. Guan adapted Donaldson’s argument in the following manner:

**Proposition 4.15.2 ([96]).** Let $(M, J)$ be a compact complex manifold and fix $\Omega$ a Kähler class on $M$. Choose a maximal compact subgroup, $G$, of the reduced automorphism group $\mathbb{H}_{\text{red}}(M, J)$ and assume that $\mathcal{M}^G_{\Omega}$ contains an extremal metric of Kähler form $\omega_0$. Denote by $E^G_{\omega_0}$ the relative $K$-energy relative to $G$ which vanishes at $\omega_0$. Then:
(i) Any extremal metric in $\mathcal{M}_{\Omega}^G$ connected to $\omega_0$ by a geodesic belongs to the $H_{\text{red}}(M,J)^G$-orbit of $\omega_0$.

(ii) For any $\omega$ in $\mathcal{M}_{\Omega}^G$ connected to $\omega_0$ by a geodesic, we have:

\begin{equation}
E_{\omega_0}^G(\omega) \geq 0,
\end{equation}

with equality if and only if $\omega$ is the $\ddbar$-Kähler form of an extremal metric.

**Proof.** The argument is quite similar to the argument for Proposition 4.15.1. Let $\omega$ be any element of $\mathcal{M}_{\Omega}^G$ connected to $\omega_0$ by a geodesic $\omega_t = \omega_0 + \ddbar \phi_t$, with $I(\phi_t) = 0$ for all $t$ in the interval $[0,1]$ (recall that $\mathcal{M}_{\Omega}^G$ is totally geodesic in $\mathcal{M}_{\Omega}$). Let $T$ be the tangent vector field along this geodesic, so that $T_{\omega_t} = \phi_t$ in $C^\infty_{\text{red}}(M,\mathbb{R}) = T_{\omega_t}\mathcal{M}_{\Omega}$. From (4.10.5), we infer

\begin{equation}
\frac{d}{dt} \sigma_{\phi_t}(T) = (D_T \sigma)(T) + \ddbar \sigma(D_T T) = (D_T \ddbar \sigma)(T)
\end{equation}

\begin{equation}
= (D_T \sigma)(T) - (D_T \zeta^G)(T).
\end{equation}

By Lemma 4.14.1, $\zeta^G$ is $\mathcal{D}$-parallel: the above can then be rewritten:

\begin{equation}
- \frac{d}{dt} \sigma_{\phi_t}(T) = \frac{d^2 E_{\omega_0}^G(\omega_t)}{dt^2} = -(D_T \sigma)(T) = 2 \int_M |D^- d\phi_t|^2 g_t v_t,
\end{equation}

cf. the proof of Proposition 4.15.1. The end of the argument is as in Proposition 4.15.1.

In terms of the Monge-Ampère equation (4.8.1) in Section 4.8, Proposition 4.15.2 can be extended as follows (the notations are as in Proposition 4.8.2):

**Proposition 4.15.3 ([60], Theorem 6.1).** Let $\Phi$ be a $G$-invariant (smooth) real function on $M \times \Sigma$, with $\Sigma = [0,T] \times S^1$. As in Proposition 4.8.2, $\Phi$ is viewed as a family $\{\phi_{t,s}\}$ of $G$-invariant functions on $M$, parametrized by $(t,s)$ in $[0,T] \times S^1$. Assume that for any $(t,s)$ in $[0,T] \times S^1$, $\phi_{t,s}$ is the relative $K$-Kähler potential with respect to $\omega_0$ of an element $(g_{t,s},\omega_{t,s})$ of $\mathcal{M}_{\Omega}^G$. We then have:

\begin{equation}
\frac{\partial^2 E^G}{\partial t^2} + \frac{\partial^2 E^G}{\partial s^2} = 2 \int_M |D^- (d\phi_t - d\phi_s)|^2 g_{t,s} v_{t,s}.
\end{equation}

In particular, $\frac{\partial^2 E^G}{\partial t^2} + \frac{\partial^2 E^G}{\partial s^2} \geq 0$, and equality holds if and only if $Z := \text{grad}_{g_{t,s}} \phi_t - J \text{grad}_{g_{t,s}} \phi_s$ is (real) holomorphic, hence belongs to $\mathfrak{h}_{\text{red}}$, for any $t,s$ in $[0,T] \times S^1$.

**Proof.** From the very definition of the relative Mabuchi $K$-energy given in Section 4.14, we infer $\frac{\partial E^G}{\partial t} = - \int_M \phi_t v_{t,s}^{G_{t,s}} v_{g_{t,s}}$, hence

\begin{equation}
\frac{\partial^2 E^G}{\partial t^2} = - \int_M \phi_t v_{t,s}^{G_{t,s}} v_{g_{t,s}} - \int_M \phi_t \frac{\partial}{\partial t} (v_{t,s}^{G_{t,s}} v_{g_{t,s}}).
\end{equation}

By using (5.2.26) in Section 5.2 below, this can be re-written:

\begin{equation}
\frac{\partial^2 E^G}{\partial t^2} = \int_M (-\phi_t + (d\phi_t, d\phi_t)_{g_{t,s}}) v_{t,s}^{G_{t,s}} v_{g_{t,s}} + 2 \int_M |D^- d\phi_t|^2 v_{t,s}^{G_{t,s}} v_{g_{t,s}}.
\end{equation}
Similarly,
\[
\frac{\partial^2 E^G}{\partial s^2} = \int_M (\Delta \tilde{\phi} + (d\tilde{\phi}, d\tilde{\phi})_{g_{t,s}}) s^G_{g_{t,s}} v_{g_{t,s}} + 2 \int_M |D^- d\tilde{\phi}|^2_{g_{t,s}} v_{g_{t,s}}.
\]

By adding (4.15.9) and (4.15.10) and by using (4.8.4), we get
\[
\frac{\partial^2 E^G}{\partial t^2} + \frac{\partial^2 E^G}{\partial s^2} = 2 \int_M (|D^- d\tilde{\phi}|^2_{g_{t,s}} + |D^- d\tilde{\phi}|^2_{g_{t,s}}) v_{g_{t,s}} + 2 \int_M (d\tilde{\phi}, d\tilde{\phi})_{g_{t,s}} s^G_{g_{t,s}} v_{g_{t,s}}.
\]

From (1.23.15), we infer
\[
\int_M (D^- d\tilde{\phi}, D^- d\tilde{\phi})_{g_{t,s}} v_{g_{t,s}} = \int_M \tilde{\phi}_t \delta D^- d\tilde{\phi} v_{g_{t,s}} = -\frac{1}{2} \int_M \tilde{\phi}_t \mathcal{L}_K \tilde{\phi}_s v_{g_{t,s}}.
\]

In the last term, we can replace \( \mathcal{L}_K \tilde{\phi}_s \), where \( K = J\text{grad}_{g_{t,s}} \) — cf. Section 2.5 — with \( \mathcal{L}_{K_G} \tilde{\phi}_s \), where \( K^G := J\text{grad}_{g_{t,s}} s^G_{g_{t,s}} \), as \( \tilde{\phi}_s \) is \( G \)-invariant. It follows that
\[
\int_M (D^- d\tilde{\phi}_t, D^- d\tilde{\phi}_s)_{g_{t,s}} v_{g_{t,s}} = -\frac{1}{2} \int_M (d\tilde{\phi}, d\tilde{\phi})_{g_{t,s}} v_{g_{t,s}} = \frac{1}{2} \langle \delta g_{t,s} (s^G_{g_{t,s}} \omega_{t,s}), \tilde{\phi}_t d\tilde{\phi}_s \rangle_{g_{t,s}}
\]
\[
= \frac{1}{2} \langle s^G_{g_{t,s}} \omega_{t,s}, d\tilde{\phi}_t \wedge d\tilde{\phi}_s \rangle_{g_{t,s}}
\]
\[
= -\frac{1}{2} \int_M (d\tilde{\phi}_t, d\tilde{\phi}_s)_{g_{t,s}} s^G_{g_{t,s}} v_{g_{t,s}}.
\]

By substituting into (4.15.11), we get (4.15.8). The last statement follows from Lemma 1.23.2.

The space \( M_0 \) is contractible and of non-positive sectional curvature, hence an (infinite-dimensional) Hadamard space. In view of the expression (4.9.1) of the curvature of the Levi Civita connection of the Mabuchi metric, it can be regarded as an (infinite dimensional) “symmetric space of non-compact type”, namely the “dual” of the group \( \text{Ham}_\omega \) of Hamiltonian symplectomorphisms of \( (M, \omega) \), hence the quotient of a (non existing) complexification of \( \text{Ham}_\omega \) by \( \text{Ham}_\omega \) (cf. [74] and Section 9 below for precise statements related to this viewpoint). In spite of that, the issue of existence/non-existence of geodesics between any two metrics in \( M_0 \) — or even between any two extremal metrics — is by no means granted a priori, cf. [133], [62].

As explained in Section 4.8, the search of geodesics between two points of \( M_0 \) essentially amounts to solving a degenerate Monge-Ampère equation on the manifold \( M \times [0,1] \times S^1 \), whith fixed boundary conditions. The main difficulty is the lack of regularity of solutions, due to the non-ellipticity of the degenerate Monge-Ampère equation. Until recently, the main progress in this direction had been accomplished by X. X. Chen [55], X. X. Chen and E. Calabi [47], who proved a general weak existence theorem for geodesics in the space of potentials \( \tilde{M}_{\omega_0} \) and a strong existence theorem for geodesics
in $\mathcal{M}_\Omega$ in the case that the first Chern class of $(M,J)$ is negative, implying the uniqueness of extremal metrics in this case.

By using Proposition 4.15.3 and an improved regularity theorem for solutions of the Monge-Ampère equation (4.8.1), the issue of uniqueness of extremal metrics within a given Kähler class — up to the action of the reduced automorphism group $H_{\text{red}}(M,J)$ — as been completely solved in this setting by X. X. Chen and G. Tian, who also proved that extremal Kähler metrics — if any — realize the minimum value of the relative K-energy on $\mathcal{M}_\Omega^G$, [60], [57], [59]. For the purpose of these notes, we here account for these results via the following statement, taken from [6]:

**Theorem 4.15.2 (X.X. Chen-G. Tian [60], [57], [59]).** For any compact complex manifold $(M,J)$ and for any Kähler class $\Omega$ on $M$, extremal metrics in $\mathcal{M}_\Omega^G$ sit in a same orbit of $H(M,J)$. Moreover, for any maximal compact subgroup $G$ of $H_{\text{red}}(M,J)$, extremal metrics in $\mathcal{M}_\Omega^G$ realize the absolute minimum of the relative K-energy $E^G_\Omega$. In particular, if $\mathcal{M}_\Omega^G$ contains an extremal metric, then $E^G_\Omega$ is bounded from below.

For the proof — which goes far beyond the scope of these notes — the reader is referred to the original papers.
The extremal Kähler cone

The extremal Kähler cone of a complex manifold $(M,J)$ is defined as the set of Kähler classes which contain an extremal Kähler metric. This chapter is mainly devoted to the LeBrun-Simanca openness theorem, according to which the extremal Kähler cone is open in the Kähler cone. We also introduced the so-called strongly extremal Kähler metrics, which are defined as the critical points of the (normalized) Calabi functional, when the latter is defined on the space of all Kähler metrics on $(M,J)$. When $M$ is of (real) dimension 4, strongly extremal Kähler metrics are characterized by the vanishing of the Bach tensor, which is the subject of Section 5.5.

5.1. LeBrun-Simanca openness theorem

In this chapter, $(M,J)$ denotes a (connected) compact complex manifold of (complex) dimension $m$. Let $H^{1,1}(M)$ be the subspace of $H^2_{dR}(M,\mathbb{R})$ of those classes which can be represented by (closed) $J$-invariant real 2-forms.

The Kähler cone, $\mathcal{K}(M,J)$, of $(M,J)$ is then an open set — possibly empty — of $H^{1,1}(M)$. A Kähler class $\Omega$ in $\mathcal{K}(M,J)$ is called extremal if it can be represented by the Kähler form of an extremal Kähler metric. The set of extremal Kähler classes in $\mathcal{K}(M,J)$ is clearly a subcone of $\mathcal{K}(M,J)$, called the extremal Kähler cone of $(M,J)$.

**Theorem 5.1.1** (C. LeBrun-S. Simanca [131] Theorem A). For any compact, connected complex manifold $(M,J)$, the extremal Kähler cone is open in $\mathcal{K}(M,J)$.

The proof of this theorem occupies the next two sections of this chapter. We essentially reproduce LeBrun-Simanca’s argument. Missing details can be found in the original papers [130] and [131] and also in [85, Section 6].

Fix an extremal Kähler class $\Omega_0$ in $\mathcal{K}(M,J)$ and an extremal Kähler metric $g_0$ of Kähler class $\omega_0$, with $[\omega_0] = \Omega_0$. Without loss of generality, we can assume that $g_0$ belongs to $\mathcal{M}^G_H$, where $G$ is a maximal connected compact subgroup of the reduces automorphism group $H_{\text{red}}(M,J)$, like in Section 4.13.

Denote by $\mathcal{M}_K$ the space of all Kähler metrics and by $\mathcal{M}^G_H$ the space of $G$-invariant Kähler metrics on $(M,J)$: a typical element of $\mathcal{M}_K$ or $\mathcal{M}^G_H$ will be indifferently viewed as a metric $g$ or as the corresponding Kähler form $\omega$, whose de Rham class can be any element in the Kähler cone $\mathcal{K}(M,J)$.

Denote by $\mathcal{H}^{1,1}_G$ the space of $J$-invariant, $g_0$-harmonic real 2-forms which are also $G$-invariant and by $\mathcal{C}^{\infty}_G$ the space of those real, $G$-invariant functions which are orthogonal to the space $P^G_{g_0}$ introduced in Section 4.13 (in particular, the elements of $\mathcal{C}^{\infty}_G$ are of zero mean value).
Following LeBrun-Simanca, we consider the map
\begin{align}
\Psi : \mathcal{M}^G_K \to \mathcal{H}^{1,1}_G \times \tilde{C}^G_G,
\end{align}
defined by
\begin{align}
\Psi(g) = (\alpha, (I - \Pi^G_{g_0})(s^G_g)),
\end{align}
for any $g$ in $\mathcal{M}^G_K$, where $\alpha$ denotes the $g_0$-harmonic representative of $[\omega - \omega_0]$, $I$ the identity operator, $\Pi^G_{g_0}$ the orthogonal projector of $L^2_G(M, \mathbb{R})$ to $P^G_{g_0}$ with respect to $g_0$, and, we recall, $s^G_g := s_g - \Pi^G_g(s_g)$ denotes the reduced scalar curvature of $g$, cf. Section 4.13. The map $\Psi$ will be referred to as the LeBrun-Simanca map.

Recall — cf. Section 4.13 — that $g$ is extremal if and only if $s^G_g$ is identically zero. Since $g_0$ is extremal, we thus have that $\Psi(g_0) = (0, 0)$. Moreover, if $g$ is close enough to $g_0$ so that $P^G_{g_0} \cap (P^G_{g_0})^\perp = \{0\}$, and if $(I - \Pi^G_{g_0})(s^G_g) = 0$, i.e. if $\Psi(g) = (\alpha, 0)$, then $s^G_g = 0$, i.e. $g$ is extremal.

The derivative, $T_{g_0} \Psi$ of $\Psi$ at $g_0$ is a linear map from $T_{g_0} \mathcal{M}^G_K = \mathcal{H}^{1,1}_G \oplus \tilde{C}^G_G$ to $T_{(0,0)} \mathcal{H}^{1,1}_G \times \tilde{C}^G_G = \mathcal{H}^{1,1}_G \oplus \tilde{C}^G_G$, hence an endomorphism of $\mathcal{H}^{1,1}_G \oplus \tilde{C}^G_G$.

The key step of the argument consists in showing that this derivative is an isomorphism in a sense which will be made precise in Section 5.3, cf. Proposition 5.3.1.

The actual computation of $T_{g_0} \Psi$ requires some additional general variational formulae which are furnished in the next Section 5.2. The end of the proof is given in Section 5.3.

### 5.2. A few variational formulae

In view of the foregoing, we need to know the derivatives of certain quantities, in particular of the reduced scalar curvature $s^G_g$, when the metric $g$ moves in $\mathcal{M}^G_K$ or $\mathcal{M}^G_K$. Since derivatives corresponding to variations within a fixed Kähler class have already been computed, it is sufficient to consider variations of the metric which are transverse to the Kähler cone, e.g. variations of the form
\begin{align}
\dot{\omega} = \alpha,
\end{align}
where $\alpha$ is harmonic with respect to $\omega$. To avoid trivial variations of the form $\dot{\omega} = k \omega$, we moreover demand that $\alpha$ be orthogonal to $\omega$, i.e. that
\begin{align}
(\alpha, \omega) = 0
\end{align}
(recall — cf. Section 4.12 — that the inner product of any harmonic 2-form $\alpha$ with the Kähler form $\omega$ is constant).

Variations as (5.2.1), with $(\alpha, \omega) = 0$, will be called trace-free harmonic variations of the metric.

**Lemma 5.2.1.** For any Kähler metric $(g, \omega)$ and any trace-free harmonic variation (5.2.1) of the metric in $\mathcal{M}_K$, the corresponding derivative at $g$ of the volume form $v_g$ is trivial:
\begin{align}
\dot{v}_g = 0.
\end{align}

**Proof.** From $v_g = \frac{\omega^m}{m!}$ we get $\dot{v}_g = \frac{\alpha \wedge \omega^{m-1}}{(m-1)!} = (\alpha, \omega) v_g$, which is equal to zero by the orthogonality condition (5.2.2).
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**Lemma 5.2.2.** For any Kähler metric \((g, \omega)\) and any trace-free harmonic variation (5.2.1) of the metric in \(M_K\), the corresponding derivative at \(g\) of the Ricci form \(\rho\) is trivial:

\[
\dot{\rho} = 0.
\]

**Proof.** Since \(\rho\) only depends upon the volume form and the complex structure — cf. Section 1.19 — Proposition 5.2.4 readily follows from Proposition 5.2.3. \(\square\)

**Lemma 5.2.3.** For any Kähler metric \((g, \omega)\) and any trace-free harmonic variation (5.2.1) of the metric in \(M_K\), the corresponding derivative at \(g\) of the scalar curvature \(s_g\) is given by:

\[
\dot{s}_g = -2(\alpha, \rho),
\]

where \(\rho\) stands for the Ricci form of \(g\).

**Proof.** Easy consequence of \(s = 2\Lambda \rho\), (5.2.4) and (3.2.11). \(\square\)

**Lemma 5.2.4.** For any Kähler metric \((g, \omega)\), for any fixed (real) holomorphic vector field \(X\) in \(\mathfrak{h}\) and for any trace-free harmonic variation (5.2.1) of the metric in \(M_K\), the corresponding derivative at \(g\) of its real potential \(f_X^g\) is given by

\[
\dot{f}_X^g = -G(\delta(\iota_X \alpha))
\]

where \(G\) stands for the Green operator, acting on the space of functions of mean value zero; equivalently, \(\dot{f}_X^g\) is of mean value zero and

\[
\Delta \dot{f}_X^g = -\delta(\iota_X \alpha).
\]

**Proof.** From (2.1.5, we infer

\[
\mathcal{L}_X \alpha = dd^c \dot{f}_X^g,
\]

hence, by using the Kähler identity (1.14.1),

\[
\Delta \dot{f}_X^g = -\Lambda dd^c \dot{f}_X^g = -\Lambda(\mathcal{L}_X \alpha) = -\Lambda(\iota_X \alpha)
\]

\[
= -[\Lambda, d]\iota_X \alpha = \delta'(\iota_X \alpha) = -\delta(\iota_X \alpha).
\]

\(\square\)

**Lemma 5.2.5.** For any Kähler metric \((g, \omega)\) and any orthogonal harmonic variation (5.2.1) of the metric in \(M_K\), the corresponding derivative at \(g\) of the Futaki character is given by

\[
\dot{\mathcal{F}}_{\Omega}(X) = -2\langle f_X^g, (\rho_H, \alpha) \rangle,
\]

for any \(X\) in \(\mathfrak{h}\) of real potential \(f_X^g\), where \(\rho_H\) stands for the \(g\)-harmonic part of the Ricci form \(\rho\).

**Proof.** From (4.12.1), we get

\[
\dot{\mathcal{F}}_{\Omega}(X) = \int_M \dot{f}_X^g v_g + \int_M f_X^g \dot{s}_g v_g = \langle f_X^g, s_g \rangle + \langle f_X^g, \dot{s}_g \rangle
\]
In this computation we several times used the fact that the volume form $\rho$ is harmonic, hence $d$- and $\delta^c$-closed, and, for further use, we deliberately broke the symmetry between $X_1, X_2$ in the final expression.}

**Lemma 5.2.7.** For any extremal Kähler metric $(g, \omega)$ in $\mathcal{M}_K^G$ and for any trace-free harmonic variation (5.2.1) of the metric in $\mathcal{M}_K^G$, the corresponding derivative at $g$ of the restriction of the Futaki-Mabuchi bilinear form $B_\Omega$ to $J_g$ is given by

\begin{equation}
\hat{B}_\Omega(X_1, X_2) = -\langle \alpha, f_g^{X_1} \, dd^c(G(f_g^{X_2})) + (G(f_g^{X_1})) \, dd^c f_g^{X_2} \rangle,
\end{equation}

for any two (fixed) elements $X_1 = \text{grad}_g f_g^{X_1}, X_2 = \text{grad}_g f_g^{X_2}$ of $J_g$.

**Proof.** From $B_\Omega(X_1, X_2) = \langle f_g^{X_1}, f_g^{X_2} \rangle$, we get $\hat{B}_\Omega(X_1, X_2) = \langle f_g^{X_1}, f_g^{X_2} \rangle + \langle f_g^{X_1}, f_g^{X_2} \rangle$ (recall that the volume form $\nu_g$ is preserved by this type of variations). By Lemma 5.2.4, this becomes

\begin{equation}
\hat{B}_\Omega(X_1, X_2) = -\langle \delta(\iota_{JX_1} \alpha), f_g^{X_2} \rangle - (f_g^{X_1}, \delta(\iota_{JX_1} \alpha))
\end{equation}

In this computation we several times used the fact that $\alpha$ is harmonic, hence $d$- and $\delta^c$-closed, and, for further use, we deliberately broke the symmetry between $X_1, X_2$ in the final expression. 

**Lemma 5.2.6.** For any Kähler metric $(g, \omega)$ in $\mathcal{M}_K^G$ and for any trace-free harmonic variation (5.2.1) of the metric in $\mathcal{M}_K^G$, the corresponding derivative at $g$ of the restriction of the Futaki-Mabuchi bilinear form $B_\Omega$ to $J_g$ is given by

\begin{equation}
\hat{B}_\Omega(X_1, X_2) = -\langle \alpha, f_g^{X_1} \, dd^c(G(f_g^{X_2})) + (G(f_g^{X_1})) \, dd^c f_g^{X_2} \rangle,
\end{equation}

for any two (fixed) elements $X_1 = \text{grad}_g f_g^{X_1}, X_2 = \text{grad}_g f_g^{X_2}$ of $J_g$.

**Proof.** From $B_\Omega(X_1, X_2) = \langle f_g^{X_1}, f_g^{X_2} \rangle$, we get $\hat{B}_\Omega(X_1, X_2) = \langle f_g^{X_1}, f_g^{X_2} \rangle + \langle f_g^{X_1}, f_g^{X_2} \rangle$ (recall that the volume form $\nu_g$ is preserved by this type of variations). By Lemma 5.2.4, this becomes

\begin{equation}
\hat{B}_\Omega(X_1, X_2) = -\langle \delta(\iota_{JX_1} \alpha), f_g^{X_2} \rangle - (f_g^{X_1}, \delta(\iota_{JX_1} \alpha))
\end{equation}

In this computation we several times used the fact that $\alpha$ is harmonic, hence $d$- and $\delta^c$-closed, and, for further use, we deliberately broke the symmetry between $X_1, X_2$ in the final expression. 

**Lemma 5.2.7.** For any extremal Kähler metric $(g, \omega)$ in $\mathcal{M}_K^G$ and for any trace-free harmonic variation (5.2.1) of the metric in $\mathcal{M}_K^G$, the corresponding derivative at $g$ of the extremal vector field $Z_{\Omega}^G$ is given by

\begin{equation}
\hat{Z}_{\Omega}^G = \text{grad}_g (\Pi_g^G (G(\alpha, dd^c s_g)) - 2(\rho, \alpha)).
\end{equation}

**Proof.** For the first part of the argument, we only assume that $g$ belongs to $\mathcal{M}_K^G$. Since $Z_{\Omega}^G$ belongs to the fixed space $J_g$ — in fact to $J_{30}$, where $30$ denotes the center of $g$ — $\hat{Z}_{\Omega}^G$ still belongs to $J_g$, hence is of the form $\hat{Z}_{\Omega}^G = \text{grad}_g (a_g^G)$ for some function $a_g^G$ in $P_G^G$. From (4.13.3) and by
using Lemmas 5.2.5 and 5.2.6 for any $X = \text{grad}_g f^X$ we then get:

$$B_\Omega(X, \hat{Z}^G_\Omega) = \langle f^X, a^G_g \rangle$$

$$= \hat{F}_\Omega(X) - \hat{B}_\Omega(X, Z^G_\Omega)$$

$$= \langle f^X, (\alpha, -2\rho_H + dd^c(G(\Pi^G_g(s)_g))) + \langle f^X, G(dd^c \Pi^G_g(s)_g), \alpha \rangle \rangle.$$

This holds for any Kähler metric $(g, \omega)$ in $\mathcal{M}^G_\Omega$. If moreover, $g$ is extremal, then $\Pi^G_g(s)_g = s_g$ and the above becomes

$$\text{for any } X \in J_g.$$ We readily infer

$$\begin{align*}
\dot{z}^G_g &= \Pi^G_g(G(\alpha, dd^c s)_g) - 2(\alpha, \rho) - 2(\alpha, \rho) \\
&= -2\Pi^G_g((\alpha, \rho)) + \Pi^G_g(G(\alpha, dd^c s)_g) - 2(\alpha, \rho).
\end{align*}$$

\[\square\]

In view of Theorem 5.1.1, the main result of this section is the following
Lemma 5.2.9. Let \((g, \omega)\) be any extremal Kähler metric in \(\mathcal{M}^G_K\). Then, for any trace-free harmonic variation (5.2.1) of the metric in \(\mathcal{M}^G_K\), the corresponding derivative of the reduced scalar curvature \(s^G_g\) at \(g\) is given by

\[ \dot{s}^G_g(\alpha) = (I - \Pi^G_g)(G((\alpha, dd^c s_g)) - 2(\alpha, \rho)). \]

For any variations of the metric in \(\mathcal{M}^G_O\), of the form \(\dot{\omega} = dd^c f\), where \(f\) is a \(G\)-invariant real function of mean value zero, the derivative of \(s^G_g\) at \(g\) is given by

\[ \dot{s}^G_g(f) = -2\delta D^{-1}df. \]

The derivative at \(g\) of the LeBrun-Simanca map \(\Psi\) is then given by

\[ T_g \Psi(\alpha, f) = (\alpha, (I - \Pi^G_g)(G((\alpha, dd^c s_g)) - 2(\alpha, \rho)) - 2\delta D^{-1}df), \]

for any \((\alpha, f)\) in \(T_g \mathcal{M}^G_K = \mathcal{H}^{1,1}_G \oplus \mathcal{C}_G^\infty\).

Proof. The reduced scalar curvature, \(s^G_g\), of any \(G\)-invariant Kähler metric \(g\) is defined by \(s^G_g = s_g - z^G_g\), cf. (4.13.2): (5.2.21) then readily follows from (5.2.16) and (5.2.5). The variations of the scalar curvature within \(\mathcal{M}_O\) is given by (3.2.6). In order to compute the variation of \(z^G_g\) within \(\mathcal{M}^G_O\), we may apply Lemma 4.5.1, as \(Z^G_G\) is now a fixed element of \(h\); we thus get

\[ \dot{z}^G_g(f) = \mathcal{L}_{Z^G_G} f = (df, dz^G_g) = (df, ds_g), \]

as \(g\) is extremal. By combining (3.2.6) and (5.2.24), we get (5.2.22). In view of (5.1.2), (5.2.23) follows (5.2.21) and (5.2.22) (notice that \(\delta \delta D^{-1}df\) is orthogonal to \(P^{G}_{G}\), so that \(\delta \delta D^{-1}df = (I - \Pi^G_g) \delta \delta D^{-1}df\)).

Remark 5.2.1. Form the proof of Lemma 5.2.9, we easily extract the following:

Lemma 5.2.10. The first variation of the reduced scalar curvature \(s^G_g\) in \(\mathcal{M}^G_O\) is given by

\[ \dot{s}^G_g(f) = -2\delta D^{-1}f + (ds^G_g, df), \]

whereas

\[ (s^G_g v_g)(f) = \delta(-2\delta D^{-1}df - s^G_g df) v_g, \]

for any \(g\) in \(\mathcal{M}^G_O\) and any \(f\) in \(T_g \mathcal{M}^G_O\).

Proof. We have: \(s^G_g = s_g - z^G_g\), whereas, by the proof of Lemma 5.2.9,

\[ \dot{z}^G_g(f) = (dz^G_g, df). \]

We then readily deduce (5.2.25) and (5.2.26) from (3.2.6) and (3.2.7).
5.3. The end of the proof of LeBrun-Simanca openness theorem

The space $\tilde{C}_G^\infty$, with its natural structure of Fréchet space, is not suitable for analysis; we thus substitute appropriate Sobolev spaces, namely, for any non-negative integer $k$, the space $\tilde{W}_G^k$ defined as the completions in $L_2^2(M, \mathbb{R})$ of $\tilde{C}_G^\infty$ for the Sobolev norm involving derivatives up to order $k$ (a precise description of the Sobolev machinery and the elliptic theory which is used below would go beyond the limits of these notes; for this we refer the reader to any one of the numerous excellent textbooks on the subject, e.g. [126] or [193]). All these spaces are Banach spaces and we denote by $\mathcal{M}_K^{G,(k)}$ the corresponding Banach manifolds.

Lemma 5.3.1. For any non-negative integer $k$, the operator $\delta \delta D^{-d}$ extends to a continuous linear operator from $\tilde{W}_G^{k+4}$ to $\tilde{W}_G^k$, which is an isomorphism.

Proof. Recall that $\delta \delta D^{-d}$ is a (formally) self-adjoint, $G$-invariant, elliptic linear fourth-order differential operator acting on $C_0^\infty(M, \mathbb{R})$. According to the standard elliptic theory, $C_0^\infty(M, \mathbb{R})$ then splits as $C_0^\infty(M, \mathbb{R}) = \ker(\delta \delta D^{-d}) \oplus \text{im}(\delta \delta D^{-d})$, where each summand is closed in $C_0^\infty(M, \mathbb{R})$ — for the Fréchet topology — and the direct sum is orthogonal with respect to the (global) inner product $\langle \cdot, \cdot \rangle$ relative to $g_0$. Since $\ker(\delta \delta D^{-d}) = (\mathbb{R} \oplus P_{g_0}^G)$ and $\tilde{C}_G^\infty(M, \mathbb{R})$ has been defined as the $L^2$-orthogonal complement of $(\mathbb{R} \oplus P_{g_0}^G)$ in $C_0^\infty(M, \mathbb{R})$, this decomposition is the same as $C_0^\infty(M, \mathbb{R}) = (\mathbb{R} \oplus P_{g_0}^G) \oplus \tilde{C}_G^\infty(M, \mathbb{R})$. In particular, the restriction of $\delta \delta D^{-d}$ to $\tilde{C}_G^\infty(M, \mathbb{R})$ is an isomorphism of $\tilde{C}_G^\infty(M, \mathbb{R})$ to itself. It follows that the natural extensions of $\delta \delta D^{-d}$ to $\tilde{W}_G^{k+4}$ are isomorphisms from $\tilde{W}_G^{k+4}$ to $\tilde{W}_G^k$, for any non-negative integer $k$.

We denote by $\Psi^{(k)}$ the extended map of Banach manifolds from $\mathcal{M}_K^{G,(k+4)}$ to $\mathcal{H}^{1,1} \times \tilde{W}_G^k$. We then have:

Proposition 5.3.1. For any non-negative integer $k$, the derivative $T_{g_0} \Psi^{(k)}$ is an isomorphism from $\mathcal{H}^{1,1} \oplus \tilde{W}_G^{k+4}$ to $\mathcal{H}^{1,1} \oplus \tilde{W}_G^k$.

Proof. It readily follows from (5.2.23) that the equation $T_{g_0} \Psi^{(k)}(\alpha, f) = 0$ is solved by the space of those pairs $(0, f)$ such that $\delta \delta D^{-d} f = 0$, when $f$ belongs to $\tilde{W}_G^{k+4}$. Since $\delta \delta D^{-d}$ is elliptic, $f$ is smooth, hence belongs to $P_{g_0}^G$. Since, moreover, $\tilde{W}_G^{k+4}$ is the completion of $\tilde{C}_G^\infty(M, \mathbb{R})$, we infer that $f \equiv 0$. This proves the injectivity of $T_{g_0} \Psi^{(k)}$. In order to check the surjectivity of $T_{g_0} \Psi^{(k)}$, choose any $(\beta, h)$ in $\mathcal{H}^{1,1} \times \tilde{W}_G^k$, and try to solve the equation $T_{g_0} \Psi^{(k)}(\alpha, f) = (\beta, h)$. This is clearly equivalent to the system: (i) $\alpha = \beta$ and (ii) $2 \delta \delta D^{-d} f = -h + (I - \Pi_{g_0}^G)((\alpha, df^s s_{g_0}) - 2(\alpha, \rho))$. Since $h$ belongs to $\tilde{W}_G^k$, whereas $\mathbb{G}((\alpha, df^s s_{g_0}))$ and $(\alpha, \rho)$ are smooth, the rhs of equation (ii) belongs to $\tilde{W}_G^k$. We then conclude by using Lemma 5.3.1.

The end of the proof of Theorem 5.1.1 goes as follows. Choose $k$ large enough so that $\mathcal{M}_K^{G,(k+4)}$ be contained in the space of Kähler metrics of regularity at least $C^4$. This means that with respect to any smooth coordinate system, in particular with respect to any smooth coordinate system arising
from a holomorphic chart determined by the complex structure $J$, the coefficients of the metric belong to the class $C^4$. Since $T_{g_0} \Psi^{(k)}$ is an isomorphism, it follows from the inverse function theorem for Banach manifolds that $\Psi^{(k)}$ determines an isomorphism from an open neighbourhood of $g_0$ in $\mathcal{M}_K^{G,(k+4)}$ to an open neighbourhood of $(0,0) = \Psi^{(k)}(g_0)$ in $\mathcal{H}_G^{1,1} \times \tilde{W}_G^k$. In particular, there exists $\epsilon > 0$ such that for any $\alpha$ in $H^1_{\mathcal{M}_G}$ with $\langle \alpha, \alpha \rangle \leq \epsilon$ there exists $g$ in $\mathcal{M}_K^{G,(k+4)}$ with the property that $\Psi^{(k)}(g) = (\alpha, 0)$. As observed in Section 5.1, if $\epsilon$ is small enough, $g$ is then an extremal Kähler metric, of regularity at least $C^4$. We conclude by using the following regularity result (the statement and the argument are taken from [131]):

**Proposition 5.3.2** ([131] Proposition 4). *Any extremal Kähler metric of regularity at least $C^4$ is smooth.*

**Proof.** By definition, a Kähler metric $g$ is extremal if the gradient $\text{grad}_g s_g$ of its scalar curvature preserves $g$, or, equivalently, if $\text{grad}_g s_g$ preserves $J$, i.e., is the real part of a holomorphic section of $T^{1,0} \mathcal{M}$, cf. Sections 1.8 and 1.23. Up to now, we only considered smooth metrics, but this makes sense if $g$ is only assumed to be of regularity $C^4$, since then $\text{grad}_g s_g$, hence also $(\text{grad}_g s_g)^{1,0}$, is of regularity $C^4$. Being holomorphic, it is actually real analytic. The dual $ds_g$ of $\text{grad}_g s_g$ with respect to $g$ is then of regularity $C^4$, and $s_g$ itself of regularity $C^5$ when expressed in terms of a holomorphic coordinate chart. Now, on any such holomorphic coordinate chart, $s_g$ has the following expression

$$(5.3.1) \quad s_g = 2\Delta \rho = \Delta_g (\log \frac{v_g}{v_0}),$$

where $v_0$ denotes the volume form of the (local) flat Kähler metric determined by the holomorphic coordinates — cf. Section 1.19 — and $v_g$ is of regularity $C^4$, as $g$ itself. By a standard bootstrapping argument using the ellipticity of $\Delta_g$ — cf. [131] for more details and references — we conclude that $g$ is smooth.

**5.4. Strongly extremal Kähler metrics**

In this chapter, we consider the (normalized) Calabi functional defined on the space of all Kähler metrics on a given compact complex manifold $(\mathcal{M}, J)$ of real dimension $n = 2m > 0$, with a special interest for the case $m = 2$, i.e., when $(\mathcal{M}, J)$ is a complex surface.

We denote: by $\mathcal{M}$ the space of all riemannian metrics on $\mathcal{M}$; by $\mathcal{M}_H$ the subspace of all $J$-hermitian (riemannian) metrics; by $\mathcal{M}_K$ the subspace of all Kähler $J$-hermitian metrics on $\mathcal{M}$; we then have

$$(5.4.1) \quad \mathcal{M}_{\Omega} \subset \mathcal{M}_K \subset \mathcal{M}_H \subset \mathcal{M},$$

where, we recall, $\mathcal{M}_{\Omega}$ denotes the space of all $J$-Kähler metrics of $\mathcal{M}$ whose Kähler class belongs to some fixed Kähler class $\Omega$.

The **normalized Calabi functional** $\mathcal{C}$ is defined on $\mathcal{M}_K$ by:

$$(5.4.2) \quad \mathcal{C}(g) = \frac{\int_{\mathcal{M}} s_g^2 v_g}{V_g^n},$$
where \( V_g = \int_M v_g \) denotes the total volume of \((M, g)\). The exponent of \( V_g \) in the denominator has been chosen in such a way that the normalized Calabi functional be scale invariant.

Following S. Simanca, a critical element of the (normalized) Calabi functional \( C \) defined on the whole of \( \mathcal{M}_K \) will be called \textit{strongly extremal Kähler metric}. Of course, a strongly extremal Kähler metric is extremal. More precisely:

**Theorem 5.4.1 (S. Simanca [176]).** Let \( g \) be a Kähler metric on a compact Kähler manifold \((M, J)\) of complex dimension \( m \geq 2 \). Denote by \( \omega \) the Kähler form, by \( \rho \) the Ricci form, by \( \rho_0 \) the trace-free — or primitive — part of \( \rho \) and by \( s \) the scalar curvature, so that \( \rho = \rho_0 + \frac{s}{2m} \omega \). Then, \( g \) is strongly extremal, i.e. critical for the restriction of \( C \) to the space \( \mathcal{M}_K \) of all Kähler metrics on \((M, J)\) metrics, if and only the following two conditions are satisfied

(i) \( g \) is extremal, i.e. \( J \text{grad}_g s \) is a Killing vector field, and

(ii) the harmonic part of \( s \rho_0 \) is zero.

If \( m = 2 \), this happens if and only if

(i) \( g \) is extremal, and

(ii) the riemannian metric \( \tilde{g} := s^{-2}g \) is Einstein on the open set \( U \) where \( s \neq 0 \).

**Proof.** Since the normalized Calabi functional is scale invariant, a metric in \( \mathcal{M}_K \) is strongly extremal if and only if it is extremal and critical for any trace-free harmonic variation \((5.2.1)\). The first statement of the theorem is then a direct consequence of \((5.2.5)\).

In order to get an easier derivation of the second statement — and also for its own sake — we now provide an alternative derivation of the first one by first considering the Calabi functional \( C \) defined on the whole of \( \mathcal{M} \): the gradient of \( C \), denoted by \( \text{grad} C \), is then a vector field on \( \mathcal{M} \), whose value at \( g \) is a symmetric bilinear form on \( M \) viewed as an element of the tangent space \( T_g \mathcal{M} \). For simplicity, the scalar curvature and the Ricci tensor will be denoted by \( s \) and \( r \), without mention of \( g \); similarly, the Levi-Civita connection of \( g \) is denoted by \( D \), whereas \( \Delta = \delta d + d \delta \) denotes the riemannian Laplace operator with respect to \( g \). The total volume will be simply denoted by \( V \). We then have:

**Lemma 5.4.1.** For any \( g \) in \( \mathcal{M} \),

\[
(\text{grad} C)(g) = 2Ds + 2\Delta s g - 2s r + \frac{s^2}{2} g - \frac{(m-2)}{2mV} \left( \int_M s^2 v_g \right) g.
\]

**Proof.** For any variation, \( \dot{g} \), of \( g \) within the space of all Riemannian metrics on \( M \), the corresponding variations, \( \dot{s}_g \) and \( \dot{v}_g \), of the scalar curvature and of the volume form are given by

\[
\dot{s}_g = \Delta (\text{tr}_g \dot{g}) + \delta \delta \dot{g} - (r, \dot{g}),
\]

and

\[
\dot{v}_g = \frac{1}{2} (\text{tr}_g \dot{g}) v_g,
\]
where $\text{tr}_g \hat{g}$ is the trace of $\hat{g}$ with respect to $g$, see e.g. [28, Theorem 1.174]: (5.4.3) follows easily.

**Remark 5.4.1.** The Calabi functional on $\mathcal{M}$ is clearly invariant under the natural action of the group of diffeomorphisms of $M$. At each point $g$ of $\mathcal{M}$, $\text{grad} C$ of $\mathcal{C}(g)$ must then be a $\delta g$-closed bilinear form on $M$, cf. [28, Chapter 2]. This can also be checked directly:

$$
\delta(Dds + \Delta s g - sr + \frac{s^2}{4} g) = \delta(Dds - \Delta(ds) + r(ds)
$$

where each line of the RHS is zero: the first one because of the general Bochner identity for 1-forms: $\Delta \alpha - \delta D \alpha = r(\alpha)$, here applied to $\alpha = ds$, the second one because of the (contracted) differential identity: $\delta r + \frac{1}{2} ds = 0$.

If $g$ belongs to $\mathcal{M}_K$, $(\text{grad} C)(g)$ splits as the sum of its $J$-invariant part, $(\text{grad} C^{J,+})(g)$, and of its $J$-anti-invariant part, $(\text{grad} C^{J,-})(g)$, with:

$$
(\text{grad} C^{J,+})(g) = 2(Dds)^{J,+} + 2\Delta s g - 2s r + \frac{s^2}{2} g - \frac{(m-2)}{2mV} (\int_M s^2 v_g) g,
$$

$$
(\text{grad} C^{J,-})(g) = 2D^- ds,
$$

At any $g$ in $\mathcal{M}_K$, the $J$-invariant part $\text{grad} C^{J,+}$ can be identified with the $J$-invariant 2-form $\gamma = (\text{grad} C^{J,+})(J\cdot,\cdot)$. Since $g$ is Kähler, $\gamma$ can be written as follows:

$$
\gamma = dd^c s + 2\Delta s \omega - 2s \rho + \frac{s^2}{2} \omega - \frac{(m-2)}{2mV} (\int_M s^2 v_g) \omega
$$

$$
= dd^c s + 2\Delta s \omega - 2s \rho_0 + \frac{(m-2)}{2m} (s^2 - \frac{1}{V} \int_M s^2 v_g) \omega
$$

$$
= dd^c s + 2\Delta s \omega - 2s \rho_0 + \frac{(m-2)}{2m} \Delta \sigma \omega,
$$

where $\rho_0$ denotes the trace-free part of $\rho$ and $\sigma$ is defined by $\sigma = \mathbb{G}(s^2 - \overline{s^2})$, where, we recall, $\mathbb{G}$ denotes the Green operator acting on the space of functions of zero mean value — cf. Section 1.17 — and $\overline{s^2} := \frac{1}{V} \int_M s^2 v_g$ denotes the mean value of $s^2$. By using the identity

$$
(\Delta f) \omega = \Delta (f \omega) = -dd^c f + \delta(df \wedge \omega),
$$

which holds for any function $f$ on any Kähler manifold — we leave the easy verification as an exercise for the reader — we get

$$
\gamma = -2s \rho_0 - dd^c s - \frac{(m-2)}{2m} dd^c \sigma + \delta((2ds + \frac{(m-2)}{2m} d\sigma) \wedge \omega).
$$

Notice that $\gamma$ can be viewed as the orthogonal projection of $(\text{grad} C)(g)$ on $T_g \mathcal{M}_H$, when elements of $\mathcal{M}_H$ are represented by their Kähler form. We then have:

**Lemma 5.4.2.** A Kähler metric $g$ in $\mathcal{M}_K$ is

(i) extremal if and only if $\gamma$ is coclosed,

(ii) strongly extremal, i.e. critical for $C|_{\mathcal{M}_K}$, if and only if $\gamma$ is coexact.
Proof. The 2-form $\gamma$ is coclosed if and only if $\delta \text{grad} C^J_g = 0$. Since $\delta \text{grad} C_g = 0$ for any metric $g$, this happens if and only if $\delta \text{grad} C^J_g = \delta D^{-1} ds = 0$. Since $M$ is compact, this is equivalent to $D^{-1} ds = 0$, i.e., $K := J \text{grad}_g s$ is Killing, hence $g$ is extremal. To prove the second assertion, it is convenient to represent metrics in $M$ by their Kähler form, so that $T g M_H \cong \Omega^{1,1} M$, the space of (real) $J$-invariant 2-forms on $M$, and $T g M_K$ is then identified with the space of closed elements of $\Omega^{1,1} M$. It follows that $\gamma$ can be regarded as the orthogonal projection of $\text{grad} C_g$ on $T g M_H$ and that its orthogonal projection on $T g M_K$ is zero, so that $g$ is critical for $C_{|M_K}$ if and only if $\gamma$ belongs to $\text{Im} \delta$.

From (5.4.10), we readily infer that the harmonic part of $\gamma$ is equal to the harmonic part of $-s \rho_0$. The first assertion of Theorem 5.4.1 then readily follows from the above lemma. This assertion can actually be formulated in a slightly more informative way as follows: an extremal Kähler metric is critical for $C_{|M_K}$ if and only if the following condition is satisfied:

$$s \rho_0 \equiv -\frac{1}{2} \alpha + \frac{(m-2)}{4m} \beta \mod \text{Im} \delta$$

or, equivalently,

$$s \rho_0 \equiv -\frac{m}{2(m-1)} (D^{-1} s) - \frac{(m-2) \beta}{4(m-1)} \mod \text{Im} \delta.$$  

This easily follows from (5.4.10) and from Lemma 5.4.2. When $m = 2$, (5.4.12) reduces to

$$s \rho_0 + (D^{-1} s) \equiv 0 \mod \text{Im} \delta.$$  

But $s \rho_0 + (D^{-1} s) \equiv 0$ is a trace-free $J$-invariant, hence antiselfdual 2-form, and any co-closed antiselfdual 2-form is clearly closed, hence harmonic. Condition (5.4.13) then simply reduces to

$$s \rho_0 + 2 \text{Hess}s = 0,$$

so that $\gamma = -\frac{3}{2} \alpha \beta$. Equivalently, since we already know that Hess is $J$-invariant, as $g$ is extremal,

$$sr_0 + 2 \text{Hess}s = 0,$$

where Hess denotes the trace-free part of the hessian. In general, for any two riemannian metrics $g, \tilde{g}$ conformally related by $\tilde{g} = \phi^{-2} g$ on a $n$-dimensional manifold, where $\phi$ is positive function, the corresponding trace-free Ricci tensors, $r, \tilde{r}$, are related by

$$r_{\phi \text{conformal}} = r_0 + (n-2) \text{Hess}(\phi) \tilde{r}. $$

cf. e.g. [28, Chapter 2]. Condition (5.4.15) then means that the conformally related metric $\tilde{g} := s^{-2} g$ is Einstein wherever it is defined, i.e. on the open set $U$ where $s$ has no zero. This completes the proof of the second assertion in Theorem 5.4.1.\[\Box\]
5.5. The four-dimensional case: the Bach tensor

When \( n = 4 \), the Calabi functional \( C : g \mapsto \int_{M} s_{g}^{2} v_{g} \) is already scale invariant and \( C_{|M_{K}} \) is actually the restriction to \( M_{K} \) of a conformally invariant functional defined on \( M \), namely the functional \( g \mapsto 24 \int_{M} |W^{+}_{g}|^{2} v_{g} \), where the selfdual Weyl tensor \( W^{+} \) is relative to the orientation induced by \( J \), for which the Kähler form of any metric in \( M_{H} \) is selfdual. This directly follows from (A.4.3) in Appendix A. Moreover, by (A.4.10), up to additive and multiplicative universal constants \( \int_{M} |W^{+}_{g}|^{2} v_{g} \) can be replaced by \( \int_{M} |W^{-}_{g}|^{2} v_{g} = \int_{M} |W_{g}|^{2} v_{g} - 12\pi^{2} \tau \) or by \( \int_{M} |W_{g}|^{2} v_{g} = \int_{M} |W^{+}_{g}|^{2} v_{g} + \int_{M} |W^{-}_{g}|^{2} v_{g} \), where \( W^{-}_{g} \) denotes the anti-selfdual Weyl tensor and \( W_{g} = W^{+}_{g} + W^{-}_{g} \) the whole Weyl tensor.

For any oriented, four-dimensional manifold \( M \), the Bach tensor, \( B^{g} \), of a riemannian metric \( g \) on \( M \) is defined as the opposite of the gradient at \( g \) of the functional
\[
(5.5.1) \quad g \mapsto W(g) := \int_{M} |W_{g}|^{2} v_{g},
\]
defined on the space \( \mathcal{M} \) of all riemannian metrics on \( M \), where \( W_{g} \) denotes the Weyl tensor of \( g \), cf. [21]. We then have:
\[
(5.5.2) \quad dW_{g}(h) = \frac{dW(g + th)}{dt}_{|t=0} = -\langle B^{g}, h \rangle_{g},
\]
for symmetric bilinear form \( h \) in \( T_{g} \mathcal{M} \). Since \( W \) is both conformally invariant and invariant under the action of the group \( \text{Diff}(M) \) of diffeomorphisms of \( M \), the Bach tensor is a co-closed, trace-free symmetric bilinear form, which is conformally covariant of weight \(-2\): \( B^{f^{-2}g} = f^{2} B^{g} \), for any positive function \( f \).

In order to express the Bach tensor, first in the general case, then in the Kähler case, we first note that the whole curvature \( R \) can written as
\[
(5.5.3) \quad R = S \wedge I + W,
\]
which is an alternative way of writing (A.1.7): here \( I \) denotes the identity and \( S \) the renormalized Ricci tensor, defined by \( S = \frac{1}{2}(r - \frac{1}{4} g) \). In (5.5.3), \( S \) is viewed as a \( T^{*}M \)-valued 1-form, the identity \( I \) as a \( TM \)-valued 1-form and the wedge product \( S \wedge I \) is the curvature-like tensor defined by
\[
(5.5.4) \quad (S \wedge I)_{X,Y} = S(X) \wedge Y + X \wedge S(Y).
\]
We then introduce the so-called Cotton-York tensor, \( C \), defined by
\[
(5.5.5) \quad C_{X,Y,Z} = -(D_{X}S)(Y,Z) + (D_{Y}S)(X,Z).
\]
The Cotton-York tensor is a three-linear form which can be viewed either as a \( T^{*}M \)-valued 2-form or as a \( \Lambda^{2}M \)-valued 1-form. As a \( T^{*}M \)-valued 2-form, \( C \) we have that
\[
(5.5.6) \quad C = -d^{D}S,
\]
where \( S \) is regarded as a \( T^{*}M \)-valued 1-form and, we recall, \( d^{D} \) denotes the formal exterior differential with respect to the Levi-Civita connection, cf. Section 1.6. As a \( \Lambda^{2}M \)-valued 1-form, \( C \) is related to the Weyl tensor \( W \) by
\[
(5.5.7) \quad C = \delta^{D}W,
\]
where $\delta^D$ denotes the formal codifferential with respect to $D$, defined as the metric formal adjoint of $d^D$ (this easily follows from the differential Binachi identity, expressed as $d^DR = 0$). We denote by $C^+$, resp. $C^-$, the selfdual, resp. anti-selfdual, part of $C$, and we then have

$$C^+ = \delta^D W^+, \quad C^- = \delta^D W^-.$$  \hfill (5.5.8)

We also recall that, on any $n$-dimensional riemannian manifold, the curvature $R$, as well as the Weyl tensor $W$, act on the space of symmetric bilinear forms by

$$R(h)(X,Y) = \sum_{i=1}^n (R_{X,e_j}Y, h(e_j)),$$  \hfill (5.5.9)

for any symmetric bilinear form $h$, also viewed as a symmetric operator, and for any auxiliary orthonormal frame $e_1, \ldots, e_n$; $R(h)(X,Y)$ is then the trace of the product of $h$ by the operator $Z \mapsto R_{X,Z}Y$; in particular, $R(g) = r$. We define $W(h)$ and $W^\pm(h)$ in a similar way. In particular, $R(h)$ and $W(h)$ are related by

$$R(h) = W(h) + \text{tr} h S + \text{tr} (h \circ S) g - S \circ h - h \circ S.$$  \hfill (5.5.10)

We then have

**Proposition 5.5.1.** The Bach tensor, $B$, of any 4-dimensional oriented manifold has the following expressions

$$B = \delta^D C + W(S)$$

$$= 2(\delta^D C^+ + W^+(S))$$

$$= 2(\delta^D C^- + W^-(S)).$$  \hfill (5.5.11)

**Proof.** We first prove the first identity in (5.5.11). For that we recall that the expression $\int_M \left( \frac{s^2}{2} - |r|^2 + |W|^2 \right)v_g$ is equal to the Euler characteristic of $M$, up to some universal positive multiplicative factor, hence is the same for all metrics on $M$. The derivative of $W$ is then readily deduced from the derivative of the functional $g \mapsto C(g) := \int_M s^2 v_g$ and $g \mapsto R(g) := \int_M |r|^2 v_g$, whose gradients are given by

$$\text{grad}_g C = 2Dds + 2\Delta s g - 2s r + \frac{s^2}{2} g,$$  \hfill (5.5.12)

and

$$\text{grad}_g R = \delta Dr + Dds + \frac{\Delta s}{2} g + \frac{|r|^2}{2} g - 2R(r),$$  \hfill (5.5.13)

cf. [28], Chapter 4. We then get

$$B^g = -\text{grad}_g W = \frac{1}{6} \text{grad}_g C - \frac{1}{2} \text{grad}_g R$$

$$= -\frac{1}{2} \delta Dr - \frac{1}{6} Dds - \frac{1}{3} s r + R(r)$$

$$+ \frac{\Delta s}{12} g + \frac{s^2}{12} g - \frac{|r|^2}{4} g.$$  \hfill (5.5.14)
On the other hand, we have

\begin{align*}
(\delta^D C)_{X,Y} &= -\sum_{j=1}^4 (D_{e_j C})_{e_j X,Y} = \sum_{j=1}^4 ((D_{e_j C}^2)_{X,Y} - (D_{e_j S})_{e_j X,Y}) \\
&= -\delta^D DS_{X,Y} - \sum_{j=1}^4 (D_{e_j X} S)_{e_j Y} + \sum_{j=1}^4 (R_{e_j X} S)_{e_j Y} \\
&= -\delta^D DS_{X,Y} + (D_X (\delta^D S))_{Y} + \sum_{j=1}^4 (R_{e_j X} S)_{e_j Y} \\
&= -\delta^D DS_{X,Y} - \frac{1}{6} (Dds)_{X,Y} + \sum_{j=1}^4 (R_{e_j X} S)_{e_j Y}.
\end{align*}

Now,

\begin{align*}
\sum_{j=1}^4 (R_{e_j X} S)_{e_j Y} &= -\sum_{j=1}^4 (S(R_{e_j X} e_j Y) + S(e_j R_{e_j X} Y)) \\
&= (\delta^D C + \frac{1}{2} (\delta^D D) - \frac{1}{2} Dds + \frac{\delta^D S}{12}) g - \frac{1}{2} r \circ r + \frac{1}{2} R(r).
\end{align*}

By comparing with (5.5.14) and by using (5.5.10) for \( h = S = \frac{r}{2} - \frac{s}{12} g \), we obtain

\begin{align*}
B^g &= \delta^D C + \frac{1}{2} R(r) + \frac{1}{2} r \circ r - \frac{1}{3} s r + \frac{s^2}{12} g - \frac{|r|^2}{4} g \\
&= \delta^D C + R(S) - \frac{1}{4} s r + \frac{1}{2} r \circ r + \frac{s^2}{12} g - \frac{|r|^2}{4} g \\
&= \delta^D C + W(S).
\end{align*}

In order to get the next two identities, it suffices to check the identity

\begin{align*}
\delta^D (C^+ - C^-) &= -(\mathcal{W}^+ - \mathcal{W}^-)(S) = -(\mathcal{W})(S).
\end{align*}

This follows from \( \delta^D = -*d^D * \), so that

\begin{align*}
\delta^D (C^+ - C^-) &= -*d^D C \\
&= *(d^D \circ d^D) S \\
&= -*(R \wedge S) \\
&= -*(W \wedge S) = -(\mathcal{W})(S).
\end{align*}

In the above computation we used the definition (5.5.6) of \( C \), as well as the expression (1.6.7) of the curvature, operating on \( C \) as a \( T^* M \)-valued 2-form; the wedge product \( R \wedge S \) then denotes the \( TM \)-valued 3-form defined by \( (R \wedge S)_{X,Y,Z} = \Theta_{X,Y,Z} R_{X,Y} S(Z) \) and it is easy to check that the Ricci-part \( S \wedge I \) contributes for nothing in this expression, as \( ((S \wedge I) \wedge S)_{X,Y,Z} = \).
$\mathcal{E}_{X,Y,Z}((S(X),S(Y))Z - S(Z,Y)S(X) + S(Z,X)S(Y) - (S(Y),S(Z))X)$, which is clearly 0; finally, it is an easy exercise to check that $(*W)(h) = *(W \wedge h)$, for any symmetric bilinear form $h$.

\textbf{Corollary 5.5.1.} The Bach tensor vanishes identically whenever one of the following three conditions is satisfied:

1. $g$ is conformally Einstein, or
2. $g$ is selfdual, i.e. $W^- \equiv 0$, or
3. $g$ is anti-selfdual, i.e. $W^+ \equiv 0$.

\begin{proof}
If $g$ is Einstein, then $r$, hence also $S$, are parallel and $C \equiv 0$; a fortiori, $\delta D C \equiv 0$, whereas $W(S) = \frac{\omega}{12} W(g) \equiv 0$; we thus get $B^g \equiv 0$ by using the first identity in (5.5.11). If $W^\pm \equiv 0$, then $C^\pm \equiv 0$ by (5.5.8) and we conclude by using the second and third identities in (5.5.11).
\end{proof}

\textbf{Corollary 5.5.2} ([67] Lemma 5, [8] Lemma 3). The Bach tensor of a Kähler surface $(M,g,J,\omega)$ has the following expression

$$B = \frac{1}{6}((2D^+ ds + s r)_0 - D^- ds),$$

where $s$ denotes the scalar curvature and $(D^+ ds - \frac{r}{2} r)_0$ stands for the trace-free part of $D^+ ds - \frac{r}{2} r$ (recall that $D^+ ds$, resp. $D^- ds$, denotes the $J$-invariant, resp. $J$-anti-invariant, part of the hessian $D ds$, cf. Section 1.23).

\begin{proof}
It is here understood that $M$ is oriented in such a way that the Kähler form $\omega$ is selfdual. Then, Corollary 5.5.2 relies on the fact that the selfdual Weyl tensor $W^+$ is given by (A.1.8), meaning that $W^+$ acts on the Kähler form $\omega$ by $W^+(\omega) = \frac{s}{6} \omega$ and on the orthogonal complement of $\omega$ in $\Lambda^+ M$ by $W^+|_{\omega^\perp} = -\frac{s}{12} I_{\omega^\perp}$; in tensorial form, $W^+$ can then be written as

$$W^+ = \frac{s}{8} \omega \otimes \omega - \frac{s}{12} \Pi^+,$$

where $\Pi^+$ denotes the orthogonal projection from $\Lambda^2 M$ to $\Lambda^+ M$. Now, $\Lambda^+ M$ splits as $\Lambda^+ M = \mathbb{R} \omega \oplus \Lambda^J M$, where $\Lambda^J M$ denotes the bundle of $J$-anti-invariant 2-forms, so that $\Pi^+$ is determined by

$$\Pi^+(X \wedge Y) = \frac{1}{2}(\omega(X,Y)\omega + X \wedge Y - JX \wedge JY)$$

(we here identify vectors and covectors as we often tacitly do in these notes whenever the metric is specified). We thus get the following explicit expression of $W^+$ in the Kähler case:

$$W^+_{X,Y}Z = \frac{s}{12} \omega(X,Y)\omega + \frac{s}{24}(-g(X,Z)Y + g(Y,Z)X + \omega(X,Z)JY - \omega(JY,Z)JX).$$

Recall that $W^+(S)_{XY}$ is defined as the trace of the product of $S$ with the operator $Z \mapsto W^+_{X,Z}Y$; we thus get:

$$W^+(S) = \frac{1}{6} S_0 = \frac{s}{12} r_0.$$
where $S_0, r_0$ denote the trace-free parts of $S, r$. From (5.5.24) we also easily obtain:

(5.5.26) \( (D_X W^+)_{Y,Z} = \frac{ds(X)}{24} (2\omega(Y, Z) \omega - Y \wedge Z + JX \wedge JZ). \)

By using (5.5.7), we thus obtain the following expression of the selfdual Cotton-York tensor $C^+$, viewed as a $TM$-valued selfdual 2-form:

(5.5.27) \[
C^+_{X,Y} = -\frac{1}{12} \omega(X, Y) \omega + \frac{1}{24} (ds(X)Y - ds(Y)X + ds(JX)JY - ds(JY)JX).
\]

We then easily infer

(5.5.28) \[
(\delta C^+)_X,Y = \frac{1}{24} (D_X ds)(Y) + \frac{1}{8} (D_{JX} ds)(JY) + \frac{1}{24} \Delta s g(X, Y),
\]

or

(5.5.29) \[
\delta C^+ = \frac{1}{6} (D^+ ds + \frac{1}{4} \Delta s g) - \frac{1}{12} D^- ds,
\]

where $D^+ ds + \frac{1}{4} \Delta s g$ is the trace-free part of $D^+ ds$ (note that $D^- ds$ is already trace-free, as a $J$-anti-invariant bilinear form). By using (5.5.25) and the second identity in (5.5.11), we finally get (5.5.21). \( \square \)

We conclude this section by rephrasing the second part of Theorem 5.4.1 in the following way:

**Theorem 5.5.1.** Let $(M, J)$ be a compact complex surface. Then, a Kähler metric $(g, J, \omega)$ on $(M, J)$ is

1. extremal, i.e. critical for the restriction of the Calabi functional to $M_\omega$, if and only if the Bach tensor $B^g$ is $J$-invariant;
2. strongly extremal, i.e. critical for the restriction of the Calabi functional to $M_K$, if and only if the Bach tensor $B^g$ is identically zero.

**Proof.** From (5.5.21) we see that the $J$-anti-invariant part of the Bach tensor $B^g$ is $-\frac{1}{6} D^- ds$: it follows that $B^g \equiv 0$ if and only if $D^- ds = 0$: this condition in turn just means that $s$ is the momentum of a hamiltonian Killing vector field, cf. Proposition 2.6.1, i.e. that the metric is extremal. By (5.5.21) again, we infer that $B^g \equiv 0$ if and only if $g$ is extremal and $(s r + 2D^+ ds)_0 \equiv 0$; since $D^- ds \equiv 0$, the latter condition can be written as $(s r + 2D ds)_0 \equiv 0$, which is exactly the condition that the metric $\tilde{g} := s^{-2} g$ be Einstein, whenever it is well-defined, i.e. on the open set, say $M_0$, where $s \neq 0$. We then conclude by using the second part of Theorem 5.4.1. \( \square \)

**Remark 5.5.1.** As observed in the beginning of this section, on any compact complex surface $(M, J)$, the Calabi functional $\mathcal{C}$ can be written as

(5.5.30) \[
\mathcal{C}(g) = \frac{1}{24} \int_M |W^+|^2 v_g = \frac{\pi^2}{2} \tau + \frac{1}{24} \int_M |W^-|^2 v_g,
\]

for any $g$ in $\mathcal{M}_K$, where, we recall, $\tau$ stands for the signature of $M$, when $M$ is oriented by the complex structure. It follows that any selfdual Kähler metric, if any, is not only critical — selfdual Kähler surfaces are Bach-flat, cf. Corollary 5.5.1 — but a (global) minimum for $\mathcal{C}$.
Remark 5.5.2. Until recently, the only known example of a compact Kähler surface whose Bach tensor is zero and whose scalar curvature $s$ is non-constant and positive everywhere (so that the Einstein metric $\tilde{g} = s^{-2}g$ is globally defined) was the first Hirzebruch surface $F_1 = \mathbb{P}^2 \# \mathbb{P}^2$, equipped with one of the $U(2)$-invariant extremal Kähler metrics constructed by E. Calabi in [45]: the Einstein metric $\tilde{g}$ then coincides with the celebrated Page metric constructed by D. Page in [158], cf. Section 10.5. Moreover, it was proved by C. LeBrun in [127] that the only compact complex surfaces which possibly admit Kähler metrics satisfying (ii), with non-constant $s$, are $\mathbb{P}^2 \# k\mathbb{P}^2$, with $k = 1, 2$ or 3. Notice that $\mathbb{P}^2 \# \mathbb{P}^2$ and $\mathbb{P}^2 \# 2\mathbb{P}^2$ are known to admit no Kähler-Einstein metric — cf. Section 6.6 — whereas the existence of extremal Kähler metrics on $\mathbb{P}^2 \# 2\mathbb{P}^2$ has recently been proved by C. Arezzo, F. Pacard and Michael Singer [15]. Similarly, $\mathbb{P}^2 \# 3\mathbb{P}^2$ admits Kähler-Einstein metrics — cf. Theorem 6.6.1 — and also extremal Kähler metrics of non-constant scalar curvature, cf. [15]. More recently, the existence of Bach-flat Kähler metrics of non-constant, everywhere positive scalar curvature — hence globally conformal to an Einstein metric, cf. above — was established by X. X. Chen, C. LeBrun and B. Weber in [58].
CHAPTER 6

The complex projective space

6.1. The complex projective space as a complex manifold

For any complex vector space $V$ of (complex) dimension $m + 1$, the (complex) projective space $\mathbb{P}(V)$ is defined as the space of complex vector subspaces of (complex) dimension 1, simply called complex lines, of $V$. The structure of complex manifold of $\mathbb{P}(V)$ is determined by the set of affine charts associated to non-zero elements of the (complex) dual $V^*$. For any non-zero $\alpha$ in $V^*$, the domain of the corresponding affine chart is the open set $\mathbb{P}(\alpha)(V) := \mathbb{P}(V) \setminus \mathbb{P}(\ker(\alpha))$, i.e. the set of those complex lines $x$ in $\mathbb{P}(V)$ which are generated by elements $u$ in $V$ such that $\alpha(u) \neq 0$. This is naturally identified with the affine subspace, $V_\alpha$, of $V$, defined by the condition $\alpha(u) = 1$, which is modeled on the kernel $\ker(\alpha)$ of $\alpha$. By choosing any element $w_\alpha$ of $\mathbb{P}(\alpha)(V) \cong V_\alpha$ as an origin, we thus get a natural bijection $v \in \ker(\alpha) \mapsto x = \mathbb{C}(w_\alpha + v) \in \mathbb{P}(\alpha)(V)$ from $\ker(\alpha)$ to $\mathbb{P}(\alpha)(V)$.

The set of affine charts defined that way makes $\mathbb{P}(V)$ into a compact complex manifold of complex dimension $m$, with respect to which the natural projection, $\pi$, from $V \setminus \{0\}$ onto $\mathbb{P}(V)$ is holomorphic.

In particular, any basis $\{e_0, \ldots, e_m\}$ of $V$ gives rise to a a full covering of $\mathbb{P}(V)$ by affine charts by successively choosing $\alpha$ equal to any element of the (algebraic) dual basis $\{e_0^*, \ldots, e_m^*\}$ of $V^*$. Via the basis $\{e_0, \ldots, e_m\}$, each element $x$ of $\mathbb{P}(V)$ is represented by its homogeneous coordinates $(u_0 : u_1 : \ldots : u_m)$ in $\mathbb{C}^{m+1}$; if, e.g. we choose $\alpha = e_0^*$, $\mathbb{P}(\alpha)(V)$ is determined by the condition $u_0 \neq 0$ and the affine space $V_\alpha$ is then the space of elements of $\mathbb{C}^{m+1}$ of the form $(1, z_1, \ldots, z_m)$, with $z_j = \frac{u_j}{u_0}$, $j = 1, \ldots, m$, whereas $\ker(\alpha)$ is the subspace $\mathbb{C}^m \subset \mathbb{C}^{m+1}$ of elements of the form $(0, z_1, \ldots, z_m)$; then, the $z_1, \ldots, z_m$ provide a holomorphic coordinate system on $\mathbb{P}(e_0^*)(V)$. We can similarly choose $\alpha = e_j^*$, for any $j = 1, \ldots, m$, and get a holomorphic coordinate system on $\mathbb{P}(e_j^*)(V)$. Since, $\mathbb{P}(V) = \bigcup_{i=0}^{m} \mathbb{P}(e_i^*)(V)$, we thus get a holomorphic atlas for $\mathbb{P}(V)$.

A (complex) 1-dimensional, resp. 2-dimensional, projective space is usually called a (complex) projective line, resp. a (complex) projective plane.

The tautological line bundle, $\Lambda$, is the complex line bundle over $\mathbb{P}(V)$ whose fiber at $x$ is $x$ itself, viewed as a (vector) complex line of $V$. By its very definition, the tautological line bundle $\Lambda$ is a subbundle of the product bundle $\mathbb{P}(V) \times V$ and comes equipped with the induced holomorphic structure: a (local) section of $\Lambda$ over some open set $U$ of $\mathbb{P}(V)$ is holomorphic if it is holomorphic a V-valued function on $U$. For any non-zero element $\alpha$ of $V^*$, the restriction of $\Lambda$ to $\mathbb{P}(V)(\alpha)$ is trivialized by the holomorphic section $\sigma_{\alpha}$ defined as follows: for any $x$ in $\mathbb{P}(V)(\alpha)$, $\sigma_{\alpha}(x)$ is the unique generator
of the complex line $x$ such that $\alpha(\sigma_\alpha(x)) = 1$. If $\alpha = e_0^*$ as above, we then have $\sigma_\alpha^*(u_0 : u_1 : \ldots : u_m) = (1, \frac{u_1}{u_0}, \ldots, \frac{u_m}{u_0}) = (1, z_1, \ldots, z_m)$.

For any $k$ in $\mathbb{Z}$, $\Lambda^k$ will denote the $k$-th (complex) tensor power of $\Lambda$ if $k$ is positive, the $(-k)$-th (complex) tensor power of the (complex) dual $\Lambda^*$ if $k$ is negative, the trivial holomorphic line bundle if $k = 0$.

**Remark 6.1.1.** The tautological line bundle $\Lambda$ will be occasionally denoted by $\mathcal{O}(-1)$, as is usually done in the literature in algebraic geometry, where holomorphic vector bundles are commonly identified with the associated sheaves of germs of holomorphic sections (in particular, $\mathcal{O}$ traditionally denotes the sheaf of germs of holomorphic functions and the associated vector bundle is then the trivial holomorphic line bundle). Accordingly, $\Lambda$ will be then denoted by $\mathcal{O}(-k)$. In particular, $\mathcal{O}(1)$ then stands for the dual tautological line bundle $\Lambda^*$. In view of Proposition 6.1.1, $\mathcal{O}(1)$ is also called the hyperplane line bundle of $\mathbb{P}(V)$.

As a subbundle of the product bundle $\mathbb{P}(V) \times V$, the tautological line bundle $\Lambda$ fits into to exact sequence

$$0 \to \Lambda \to \mathbb{P}(V) \times V \to Q \to 0,$$

where $Q$ is defined as the vector bundle over $\mathbb{P}(V)$ whose fiber at $x$ is the quotient $V/x$. On the other hand, at each point $x$ of $\mathbb{P}(V)$, the tangent space $T_x\mathbb{P}(V)$ admits the following natural identification

$$T_x\mathbb{P}(V) = \text{Hom}(x, V/x) = x^* \otimes V/x,$$

where $\text{Hom}(x, V/x)$ denotes the space of $\mathbb{C}$-linear homomorphisms from the complex line $x$ to the quotient $V/x$. This can be shown as follows. For any $\alpha$ in $V^*$ such that $\alpha|_x \neq 0$, denote by $P_\alpha$ the (complex) hyperplane $\ker \alpha$ in $V$. We thus have $V = x \oplus P_\alpha$ and $\mathbb{P}^{(\alpha)}(V)$ is identified with the complex vector space $\text{Hom}(x, P_\alpha)$ by representing any $y$ in $\mathbb{P}(V)^{(\alpha)}$ by its graph with respect to the above splitting of $V$, namely $y = \{u + A_y(u) | u \in x\}$, for some well-defined $A_y$ in $\text{Hom}(x, P_\alpha)$. We thus get: $T_x\mathbb{P}(V) = T_x\mathbb{P}^{(\alpha)}(V) = \text{Hom}(x, P_\alpha = \text{Hom}(x, V/x)$, via the identification $P_\alpha = V/x$, deduced from the above splitting of $V$. It is easy to check that the resulting isomorphism $T_x\mathbb{P}(V) = \text{Hom}(x, V/x)$ is independent of the choice of $\alpha$. Alternatively, any curve $x_t$ in $\mathbb{P}(V)$, with $x_t = x$, can be lifted to a curve $u_t$ in $V \setminus 0$, with $u_0 = u$ any non-zero element of $x$; then $x_0$ is entirely determined by $u_0 \mod x$. It follows that $T\mathbb{P}(V) = \Lambda^* \otimes Q$, hence fits into the exact sequence

$$0 \to 1 \to \Lambda^* \otimes V \to T\mathbb{P}(V) \to 0$$

obtained by tensoring (8.5.1) by $\Lambda^*$; here, $1 = \Lambda^* \otimes \Lambda$ stands for the trivial bundle $\mathbb{P}(V) \times \mathbb{C}$ and $\Lambda^* \otimes V$ is (non canonically) isomorphic to $(m+1)$ copies of $\Lambda^*$.

**Remark 6.1.2.** From (6.1.2), we easily infer the following isomorphism:

$$K^*(\mathbb{P}(V)) = \mathcal{O}(m+1),$$

where, we recall — cf. Section 1.19 — $K^*(\mathbb{P}(V)) = \Lambda^m(T\mathbb{P}(V))$ denotes the anti-canonical line bundle of $\mathbb{P}(V)$. Indeed, from (6.1.2) we infer that the fiber at $x$ of $K^*(\mathbb{P}(V))$ is canonically identified with $(x^*)^m \otimes \Lambda^m(V/x)$,
whereas $\Lambda^m(V/x)$ is canonically identified with $x^* \otimes \Lambda^{m+1}V$ (equivalently, $\Lambda^{m+1}V = x \otimes \Lambda^m(V/x)$ for any complex line $x$ in $V$).

**Proposition 6.1.1.** For any positive integer $k$, the space $H^0(\mathbb{P}(V), \mathcal{O}(k))$ of holomorphic sections of $\mathcal{O}(k)$ is naturally isomorphic to the complex $k$-th symmetric tensorial power $\bigodot^k V^*$, whereas $H^0(\mathbb{P}(V), \mathcal{O}(-k)) = \{0\}$.

**Proof.** For simplicity we give a (sketchy) argument for $k = 1$ only. Any holomorphic section $\sigma$ of $\Lambda^*$ determines a holomorphic function $\tilde{\sigma}$ on $V \setminus 0$ defined by $\tilde{\sigma}(u) = \sigma_x(u)$ where $x = \pi(u)$. By Hartogs' theorem — cf. e.g. [104], [155] — $\tilde{\sigma}$ extends to a holomorphic function, still denoted $\tilde{\sigma}$, defined on $V$, with $\tilde{\sigma}(0) = 0$. Moreover, $\tilde{\sigma}$ satisfies the identity $\tilde{\sigma}(\lambda u) = \lambda \tilde{\sigma}(u)$, for any $u$ in $V$, any $\lambda$ in $\mathbb{C}$: it thus coincides with its derivative at 0 and is then a linear form on $V$. Conversely, any element $\alpha$ of $V^*$ determines a holomorphic section of $\Lambda^*$, say $\tilde{\alpha}$, defined by $\tilde{\alpha}(x) = \alpha_x$, where $\alpha_x$ denotes the restriction of $\alpha$ to the complex line $x$. We then get $H^0(\mathbb{P}(V), \Lambda^{-1}) = V^*$. The argument for $H^0(\mathbb{P}(V), \Lambda^{-k}) = \bigodot^k V^*$ is quite similar. The second assertion $H^0(\mathbb{P}(V), \Lambda^k) = \{0\}$ follows by the following general simple fact: A holomorphic line bundle $L$ and its dual $L^*$ over a compact, connected, complex manifold $M$ cannot both admit a non-trivial holomorphic section, unless they are both (holomorphically) trivial. Indeed, if $\sigma$ is a non-trivial holomorphic section of $L$ and $\alpha$ a holomorphic section of $L^*$, then $\alpha(\sigma)$ is a holomorphic function on $M$ hence is constant, as $M$ is compact; if $L$ is not trivial, $\sigma$ has zeros, so that this constant is 0; since $\sigma$ is non-trivial it has no zeros on a dense open set of $M$, on which $\alpha$ vanishes necessarily as $\alpha(\sigma) \equiv 0$: $\alpha$ is then identically 0. □

**Proposition 6.1.2.** The group $\text{Aut}(\mathbb{P}(V))$ of complex automorphisms of $\mathbb{P}(V)$ is naturally isomorphic to the projectivized complex linear group $PG\ell(V) = \text{GL}(V)/\mathbb{C}^*$.

**Proof.** Any element $\gamma$ of $GL(V)$ tautologically induces a holomorphic automorphism, say $\tilde{\gamma}$, of $\mathbb{P}(V)$, by setting $\tilde{\gamma}(x) = \pi(\gamma(u))$, for any non-zero $u$ in $x$; we thus get a homomorphism from $GL(V)$ to $\text{Aut}(\mathbb{P}(V))$, whose kernel is clearly the subgroup $\mathbb{C}^* I$, where $I$ denotes the identity of $GL(V)$. We thus get an injective homomorphism from $PG\ell(V)$ to $\text{Aut}(\mathbb{P}(V))$. In order to show that this homomorphism is surjective, i.e. that $PG\ell(V)$ is actually equal to $\text{Aut}(\mathbb{P}(V))$, we use the fact that $\mathbb{P}(V)$ admits no other holomorphic line bundle than the $\Lambda^k$ already mentioned, for $k \in \mathbb{Z}$. This fact in turn follows from the fact that $H^2(\mathbb{P}(V), \mathbb{Z}) = \mathbb{Z}$ — so that (topological) complex line bundles are parametrized by their degree, equal to $-k$ for $\Lambda^k$ — and that $\mathbb{P}(V)$ is simply-connected — as a quotient of the sphere $S^{2m+1}$ by a free action of the circle $S^1$, cf. Section 6.2 — so that $\Lambda^k$ has no other holomorphic structure than the standard one (for a justification of these assertions we refer the reader to e.g. [61] or [93]). For any $\Phi$ in $\text{Aut}(\mathbb{P}(V))$, $\Phi^*(\Lambda)$ is a holomorphic line bundle of degree $\pm 1$, as $\Phi$ is a diffeomorphism, in fact $-1$, as $\Phi^*(\Lambda)$, like $\Lambda$, has no non-zero global holomorphic section; it follows that $\Phi^*(\Lambda)$ is isomorphic to $\Lambda$ as a holomorphic line bundle. For any chosen holomorphic isomorphism of $\Phi^*(\Lambda)$ with $\Lambda$ we then get a biholomorphic map, say $\tilde{\Phi}$, from $V \setminus \{0\}$ to itself, which commutes with $\pi$; by Hartogs' theorem
— cf. the proof of Proposition 6.1.1 — \( \Phi \) extends to a biholomorphic map from \( V \) to itself; moreover, since \( \Phi \) satisfies \( \Phi(\lambda u) = \lambda \Phi(u) \), for any \( u \) in \( V \), any \( \lambda \) in \( \mathbb{C} \), we infer as in the proof of Proposition 6.1.1, that \( \Phi \) is actually a linear map, hence an element of \( GL(V) \); since \( \text{Aut}(L) = \mathbb{C}^* \) — as \( \mathbb{P}(V) \) is compact — for any other chosen isomorphism \( \Phi^*(\Lambda) \cong \Lambda \), the resulting \( \Phi \) is a multiple of the former one by an element of \( \mathbb{C}^* \): thus we get a well defined element of \( PG\ell(V) \), whose image in \( \text{Aut}(\mathbb{P}(V)) \) is obviously \( \Phi \). □

**Remark 6.1.3.** From Proposition 6.1.2 we infer that the Lie algebra \( \mathfrak{h} \) of (real) holomorphic vector fields on \( \mathbb{P}(V) \), which is the Lie algebra of \( \text{Aut}(\mathbb{P}(V)) \), is naturally isomorphic to the Lie algebra \( gl(V)/\mathbb{C}I \), the quotient of the Lie algebra \( gl(V) \) of \( \mathbb{C} \)-linear endomorphisms of \( V \) by the the space generated by the identity, \( I \), of \( V \), which is the center of \( gl(V) \). For any \( a \) in \( gl(V) \), the corresponding vector field \( Z^a \) is given by

\[
Z^a(x) = a|_x \mod x,
\]

for any \( x \) in \( \mathbb{P}(V) \) (we here use the identification \( T_x \mathbb{P}(V) = \text{Hom}(x, V/x) \), cf. (6.1.2), and \( a|_x \) denotes the restriction of \( a \) to the complex line \( x \); we readily check that \( Z^a \equiv 0 \) if \( a = I \), whereas \( a \) can be recovered from \( Z^a \), mod \( I \).

Let \( \{ e_0, \ldots , e_m \} \) be any basis of \( V \), so that \( a \) has the following matrical expression:

\[
a(e_j) = \sum_{k=0}^m a_{kj} e_k
\]

and consider any affine chart determined by any element \( e_j^* \) of the dual basis of \( V^* \), as explained in the beginning of this section; to simplify notation, choose \( e_0^* \): we then have \( \mathbb{P}(e_0^*)(V) = \{ (u_0 : \ldots : u_m) | u_0 \neq 0 \} \), which is identified with \( \mathbb{C}^m = \ker(e_0^*) \subset V = \mathbb{C}^{m+1} \) by setting \( z_1 = \frac{u_1}{u_0}, \ldots , z_m = \frac{u_m}{u_0} \). Then, the restriction of \( Z^a \) to \( \mathbb{P}(e_0^*)(V) \), viewed as a vector field on \( \mathbb{C}^m \), has the following expression:

\[
Z^a = \sum_{j=1}^m (a_{j0} + \sum_{k=1}^m a_{jk} z_k - z_j (a_{00} + \sum_{k=1}^m a_{0k} z_k)) e_j.
\]

Indeed, any \( x \) in \( \mathbb{P}(e_0^*)(V) = \mathbb{C}^m \), of affine holomorphic coordinates \( z_1, \ldots , z_m \), is generated by \( u_x := e_0 + \sum_{j=1}^m z_je_j \), whose image by \( a \) is \( a(u_x) = a_{00} e_0 + \sum_{k=1}^m a_{0k} e_k + z_j (a_{0j} e_0 + \sum_{k=1}^m a_{kj} e_k) \); by projecting \( a(u_x) \) on \( \mathbb{C}^m = \ker(e_0^*) \) along \( x \), we readily get the rhs of (6.1.6). Notice that the rhs of (6.1.6) can ve indifferently regarded as a (real) holomorphic vector field — our usual viewpoint — or as a holomorphic (complex) vector field of type \((1,0)\), when it is better written as \( Z^a = \sum_{j=1}^m (a_{j0} + \sum_{k=1}^m a_{jk} z_k - z_j (a_{00} + \sum_{k=1}^m a_{0k} z_k)) \partial/\partial z_j \).

The (complex) Lie algebra \( \mathfrak{h} = \mathfrak{sl}(V) \) is reductive, even semi-simple, and then provides no obstruction to the existence of Kähler-Einstein metrics on complex projective spaces. These actually exist and are the so-called Fubini-Study metrics described in the next section.

### 6.2. The Fubini-Study metric

Any basis of \( V \) identifies \( V \) with \( \mathbb{C}^{m+1} \) and \( \mathbb{P}(V) \) with the *standard projective space* \( \mathbb{P}^m := \mathbb{P}(\mathbb{C}^{m+1}) \).

Via the standard hermitian inner product \( \langle \cdot , \cdot \rangle \) of \( \mathbb{C}^{m+1} \), \( \mathbb{P}^m \) comes equipped with a 1-parameter family of canonical Kähler metrics defined
as follows. Denote by $S^{2m+1}$ the unit sphere in $\mathbb{C}^{m+1}$ and by $\hat{\pi}$ the restriction of $\pi$ to $S^{2m+1}$. Then, $\hat{\pi}$ makes $S^{2m+1}$ into a principal $S^1$-bundle over $\mathbb{P}^m$, for the action of the $S^1$ determined by the usual diagonal action of $S^1$ on $\mathbb{C}^{m+1}$. The standard euclidean inner product $\Re(\cdot, \cdot)$ of $\mathbb{C}^{m+1}$ induces a riemannian metric on $S^{2m+1}$, which is the standard metric of constant sectional curvature +1, denoted by $g_0$.

If $h$ denotes the fiberwise hermitian structure of the tautological line bundle $\Lambda$ induced by the inner product of $\mathbb{C}^{m+1}$. $S^{2m+1}$ is then the associated $S^1$-principal bundle of “unit frames”. Any $x$ in $\mathbb{P}^m$ is then represented by a unit element $u$ in $S^{2m+1}$, which is well-defined up to the action of $S^1$: $u \rightarrow e^{it}u$. Let $T$ be the generator of the action of $S^1$ on $S^{2m+1}$ and denote by $\eta$ the dual 1-form with respect to $g_0$. A vector in $T_u S^{2m+1}$ is called horizontal if its belongs to the kernel of $\eta$. Then, for any $u$ in $S^{2m+1}$ over $x$, any vector $X$ in $T_u \mathbb{P}^m$ is uniquely represented by a unique horizontal vector $\tilde{X}$ in $T_u S^{2m+1}$ which projects to $X$ by $\hat{\pi}_s$. For any positive real number $c$, we then define an almost hermitian structure $(g_{FS,c}, J_{FS}, \omega_{FS,c})$ on $\mathbb{P}^m$ by setting:

\[
g_{FS,c}(X,Y) = \frac{4}{c} g_0(\tilde{X}, \tilde{Y}),\]

\[
J_{FS}X = \hat{\pi}_s(i\tilde{X}),\]

\[
\omega_{FS,c}(X,Y) = \frac{4}{c} g_0(i\tilde{X}, \tilde{Y}).\]

It is easily checked that the above expressions are independent of the choice of $u$ in $\hat{\pi}^{-1}(x)$ and that the almost-structure $J_{FS}$ coincides with the almost-structure induced by the natural complex structure of $\mathbb{P}^m$ defined via the affine charts. In particular, $J_{FS}$ is integrable. Moreover we easily check that

\[
\hat{\pi}^* \omega_{FS,c} = \frac{2}{c} \, d\eta.
\]

It follows that $\omega_{FS,c}$ is closed. By Proposition 1.1.1 the almost hermitian structure defined by (6.2.1) is then a Kähler structure, called the Fubini-Study metric of $\mathbb{P}(V)$ of parameter $c$ induced by the basis $s$ (later on in this section, the parameter $c$ will be indentified with the (constant) holomorphic sectional curvature of $\omega_{FS,c}$).

**Remark 6.2.1.** The Fubini-Study metric of parameter $c$ induced by any basis of $V$ only depends on the hermitian inner product of $V$ determined by this basis; moreover, it remains unchanged when the basis is replaced by any homothetic basis. Denote by $\mathcal{B}(V)$ the space of all basis of $V$ and by $\mathcal{B}_0(V)$ the quotient $\mathcal{B}(V)/\mathbb{C}^*$. Then, the natural (right) action of the special linear group $SL(m+1, \mathbb{C})$ on $\mathcal{B}_0(V)$ is transitive and, for any chosen base-point in $\mathcal{B}(V)$, the space, $\mathcal{F}(V)$, of all Fubini-Study metrics of $\mathbb{P}(V)$ of any chosen parameter $c$ is then identified with the quotient $SL(m+1, \mathbb{C})/SU(s_0)$, where $SU(s_0) \cong SU(m+1)$ stands for the special unitary group of the hermitian inner product determines by $s_0$. Since $SU(m+1)$ is a maximal connected compact Lie subgroup of $SL(m+1, \mathbb{C})$, $\mathcal{F}(V) \cong SL(m+1, \mathbb{C})/SU(m+1)$ has the structure of a riemannian symmetric space of non-compact type, of dimension $m(m+2)$. In particular, $\mathcal{F}(V)$ is diffeomorphic to $\mathbb{R}^{m(m+2)}$ (for more information on symmetric spaces we refer the reader to e.g. [98]).
In order to check that the Fubini-Study metric is Kähler-Einstein, there is no need to compute the whole curvature: we can use instead the direct definition of the Ricci form \( \rho \) as the curvature of the Chern connection of the anti-canonical line bundle \( K_{\mathbb{P}^m}^{-1} \) which we already identified with \( \mathcal{O}(-(m + 1)) = \Lambda^{-(m-1)} \), cf. Remark 6.1.2. Indeed, the 1-form \( \eta \) is nothing else than the connection 1-form on the principal bundle \( S^{2m+1} \) of the Chern connection \( \nabla^\Lambda \) of \( \Lambda \) and (6.2.2) then simply means that the curvature form \( \rho^\Lambda \) of \( \nabla^\Lambda \), as defined in Section 1.19, is related to \( \omega_{\text{FS}} \) by

\[
\omega_{\text{FS},c} = -\frac{2}{c} \rho^\Lambda.
\]

By (1.7.3), we infer that

\[
\tilde{\pi}^* \omega_{\text{FS},c} = \frac{1}{c} \dd c \log r^2
\]

where \( \tilde{\pi} \) denotes the projection of \( \tilde{\Lambda} := \Lambda \setminus \Sigma_0 \) onto \( \mathbb{P}^m \).

**Remark 6.2.2.** On the affine chart determined by \( e_0^* \), where the affine holomorphic coordinates on \( \mathbb{P}^{\tilde{E}}(\mathbb{C}^{m+1}) \) are denoted by \( z_1, \ldots, z_m \), cf. Section 6.1, the map \( s : (z_1, \ldots, z_m) \mapsto (1, z_1, \ldots, z_m) \) is a holomorphic trivialization of the tautological line bundle \( \Lambda \) over \( \mathbb{P}^{\tilde{E}}(\mathbb{C}^{m+1}) \), whose square norm is equal to \( 1 + \sum_{j=1}^{m} |z_j|^2 \). It then follows from (1.7.2) that the restriction of \( \omega_{\text{FS},c} \) to \( \mathbb{P}^{\tilde{E}}(\mathbb{C}^{m+1}) \) has the following expression:

\[
\omega_{\text{FS},c} = \frac{1}{c} \dd c \log (1 + \sum_{j=1}^{m} |z_j|^2).
\]

The Ricci form \( \rho_{\text{FS}} \) is the curvature form of the anti-canonical line bundle \( K_{\mathbb{P}^m}^{-1} = \Lambda^{-(m+1)} \), equipped with the induced hermitian structure; it is then given by

\[
\rho_{\text{FS}} = -(m + 1) \rho^\Lambda = \frac{c}{2} (m + 1) \omega_{\text{FS}}.
\]

It follows that the Fubini-Study metric is Kähler-Einstein, of (constant) scalar curvature given by

\[
s_{\text{FS},c} = cm(m + 1).
\]

On the other hand, it is easy to compute the whole curvature, \( R_{\text{FS},c} \), of \( g_{\text{FS},c} \), and we then get:

**Proposition 6.2.1.** For any \( c > 0 \), the curvature \( R_{\text{FS},c} \) of the corresponding Fubini-Study metric is given by

\[
R_{\text{FS},c}^{X,Y} Z = \frac{c}{4} ((X, Z)Y - (Y, Z)X + (JX, Z)JY - (JY, Z)JX + 2(JX, Y)JZ),
\]

for any three vector fields \( X, Y, Z \). For any tangent 2-plane \( P \), generated by on orthonormal pair \( X, Y \), the sectional curvature \( K(P) \) of \( P \) is then given by

\[
K(P) = \frac{c}{4} (1 + 3 \cos^2 \theta),
\]
where $\theta$, called the holomorphy angle of $P$, is defined by $\cos \theta = \omega_{F,c}(X,Y)$. In particular, the holomorphic sectional curvature is constant, equal to $c$, and the Bochner tensor — see Section (1.19) — vanishes identically.

Proof. For any vector fields $X,Y$ on $\mathbb{P}^m$, represented by their horizontal lifts $\tilde{X},\tilde{Y}$, the horizontal lift, $D_XY$, of the covariant derivative $D_XY$ relative to the Levi-Civita connection of the Fubini-Study metric — the same for all values of $c$ — is equal the horizontal projection of $D^S_X\tilde{Y}$ along $T$, where $D^S$ denotes the Levi-Civita connection of the sphere $(S^{2m+1},g_0)$ (easy verification). We also easily check that $D^S\tilde{T} = 0$; readily follows from these observations. The rest of the Proposition follows (of course, (6.2.6) also readily follows from (6.2.8)).

**Proposition 6.2.2.** The connected isometry group $K(\mathbb{P}^m,g_{FS})$ is the subgroup $PU(m+1) = U(m+1)/\mathbb{C}^* = SU(m+1)/\mu_{m+1}$ of $PG\ell(m+1,\mathbb{C}) = Gl(m+1,\mathbb{C})/\mathbb{C}^*$, where $\mu_{m+1} = \mathbb{C}^* \cap SU(m+1)$ stands for the group of $(m+1)$-th roots of 1, realized as the center of $SU(m+1)$.

Proof. By its very definition, the unitary group $U(m+1)$ preserves the natural hermitian inner product of $\mathbb{C}^{m+1}$; the projectivized unitary group $PU(m+1)$, as a subgroup of $PG\ell(m+1,\mathbb{C}) = Aut(\mathbb{P}(V))$, is then a subgroup of $K(\mathbb{P}^m,g_{FS})$: in fact, the two groups coincide, as $PU(m+1)$ is a maximal compact subgroup of $PG\ell(m+1,\mathbb{C})$.

**Proposition 6.2.3.** All Kähler-Einstein metrics on $\mathbb{P}^m$ are of (positive) constant holomorphic sectional curvature and are isomorphic to a Fubini-Study metric by the action of an element of $Aut(\mathbb{P}^m)$.

Proof. We first notice that $\mathbb{P}^m$ admits no Kähler-Einstein metric with negative or zero scalar curvature, since its first Chern class is positive, cf. Section 1.21. By Theorem 3.6.2, the (identity component of the) isometry group of any Kähler-Einstein metric with positive scalar curvature is a maximal connected compact subgroup of $Aut(\mathbb{P}^m)$ and is then conjugate in $Aut(\mathbb{P}^m)$ to $SU(m+1)/\mu_{m+1}$. The Kähler-Einstein metric is then the image by an element of $Aut(\mathbb{P}^m)$ of a Kähler-Einstein metric whose isometry group is $SU(m+1)/\mu_{m+1}$. The latter is then equal to a Fubini-Study metric. Indeed, the Ricci form of any Kähler metric on a given complex manifold is entirely determined by its volume form — cf. Section 1.19 — and, up to rescaling, there is only one $SU(m+1)/\mu_{m+1}$-invariant volume form, as $SU(m+1)/\mu_{m+1}$ acts transitively on $\mathbb{P}^m$.

**Remark 6.2.3.** The volume of $\mathbb{P}^m$ with respect to the Fubini-Study metric of holomorphic sectional curvature $c$ is equal to $\frac{1}{m!} (\frac{2\pi}{c})^m$. This follows readily from (6.2.3) and the fact that the Chern class of the dual tautological line bundle $\Lambda^* = \mathcal{O}(1)$ is the generator of $H^2(\mathbb{P}^m,\mathbb{Z})$, so that $\int_{\mathbb{P}^m} (-\frac{2\pi}{c})^m = 1$.

**Remark 6.2.4.** By choosing any base-point $x_0$, say $x_0 = (1:0:\ldots:0)$, in $\mathbb{P}^m$, the latter, equipped with a Fubini-Study Kähler structure of constant holomorphic sectional curvature $c$, can be identified with the quotient of $SU(m+1)/\mu_{m+1}$ by the isotropy group of $x_0$ which is clearly the
group \( S(U(1) \times U(m))/\mu_{m+1} := SU(m+1) \cap (U(1) \times U(m))/\mu_{m+1} \). We thus identify \( \mathbb{P}^m \) with the quotient \( SU(m+1)/S(U(1) \times U(m)) \), which is a hermitian symmetric space of compact type, cf. e.g. [98]. The reader is invited to explore and retrieve the main properties of the Fubini-Study metric in this setting. Any riemannian symmetric space of compact type has an associated riemannian symmetric space of non-compact type [98]. In the current case, this is the standard complex hyperbolic space \( \mathbb{CH}^m = SU(m,1)/S(U(1) \times U(m)) \), where \( SU(m,1) \) stands for the unitary group of \( \mathbb{C}^{1,m} := \mathbb{C} \oplus \mathbb{C}^m \), equipped with the Lorentzian hermitian form, \( (\cdot,\cdot)_{1,m} \) obtained by reversing the sign of the standard hermitian form on the factor \( \mathbb{C}^m \). As a complex manifold, the complex hyperbolic space \( \mathbb{CH}^m \) can be realized as the space of timelike complex lines in \( \mathbb{C}^{1,m} \), defined as the (vector) complex lines on which \( (\cdot,\cdot)_{1,m} \) is positive definite. The construction of the dual Fubini-Study metric can then be done as before, by simply replacing the former unit sphere \( S^{2m+1} \) by the Lorentzian unit sphere \( S^{1,2m}_1 \) in \( \mathbb{C}^{1,m} \): this “unit sphere” \( S^{1,2m}_1 \) is no longer a topological sphere, but is still a \( S^1 \)-principal bundle over \( \mathbb{CH}^m \) and comes equipped with a \( S^1 \)-invariant lorentzian metric of signature \((2m,1)\), induced by \(- (\cdot,\cdot)_{1,m} \), which is positive definite on the horizontal space determined by the \( S^1 \)-action. The dual Fubini-Study metric is then the induced metric on \( \mathbb{CH}^m = S^{1,2m}_1/S^1 \), as defined by (6.2.1). It is easily checked that the resulting Kähler metric is of constant (negative) holomorphic sectional curvature, equal to \(-c\).

Remark 6.2.5. A variant of the construction of the Fubini-Study and the dual Fubini-Study metrics relies on the natural isomorphism

\[
(6.2.10) \quad T_x \mathbb{P}^m = \text{Hom}(x, \mathbb{C}^{m+1}/x),
\]

for any \( x \) in \( \mathbb{P}^m \) (cf. Section 6.1). If \( \mathbb{C}^{m+1} \) is equipped with its standard (positive definite) hermitian product, then \( \mathbb{C}^{m+1}/x \) is identified with the hermitian orthogonal complement, \( x^\perp \) of \( x \) in \( \mathbb{C}^{m+1} \), which recovers a definite positive hermitian product, as well as the complex line \( x \) itself. It follows that \( \text{Hom}(x, \mathbb{C}^{m+1}/x) = x^* \otimes x^\perp \) admits a positive definite hermitian inner product, which makes \( \mathbb{P}^m \) into a hermitian complex manifold. It is easy to check that this hermitian structure coincides with the Fubini-Study Kähler structure defined by (6.2.1), for the value \( c = 4 \) of the holomorphic sectional curvature. The dual Fubini-Study can be defined similarly, when \( \mathbb{CH}^m \) is holomorphically embedded in \( \mathbb{P}^m \) as the open set of timelike elements, when \( \mathbb{C}^{m+1} \) is now equipped with a lorentzian hermitian inner product of signature \((1,m)\), cf. Remark 6.2.4. If now, for any \( x \) in \( \mathbb{CH}^m \), \( x^\perp \) denotes the orthogonal complement of \( x \) in \( \mathbb{C}^{m+1} \) with respect to the lorentzian hermitian inner product, then the latter restrict to a negative definite hermitian inner product on \( x^\perp \) and to a positive definite hermitian inner product on \( x \) itself. By changing the sign, we eventually get a positive definite hermitian inner product on \( x^* \otimes x^\perp = T_x \mathbb{CH}^m \), hence a hermitian structure on \( \mathbb{CH}^m \), which is the dual Fubini-Study Kähler structure of holomorphic sectional curvature \(-4\).
6.3. The complex projective space as a (co)adjoint orbit

We start this section by recalling some well-known general facts concerning hamiltonian actions and momentum maps. Let \((M,\omega)\) be a symplectic manifold acted on — on the left — by a Lie group \(G\) and suppose that this action preserves the symplectic form \(\gamma^*\omega = \omega\) for any \(\gamma\) in \(G\). For any \(a\) in the Lie algebra \(g\) of \(G\), the induced vector field \(\hat{a}\) is the vector field on \(M\) whose flow is given by the action of the 1-parameter subgroup \(\exp(ta)\) of \(G\) generated by \(a\), where \(\exp\) stands for the exponential map from \(g\) to \(G\); we then have:

\[
\hat{a}(x) = \frac{d}{dt}|_{t=0} \exp(ta) \cdot x
\]

for any \(x\) in \(M\). Note that

\[
\begin{align*}
\hat{a}, \hat{b} &= -[a, b], \\
\mu_{[a, b]}(x) &= -\omega_x(\hat{a}, \hat{b}),
\end{align*}
\]

for any two elements \(a, b\) of \(g\), as the action of \(G\) is a left action. From (6.3.1) and (1.4.1) we infer that

\[
L_{\hat{a}} \omega = \frac{d}{dt}|_{t=0} \exp(ta)^* \omega = 0.
\]

Each \(\hat{a}\) is then a symplectic vector field, as defined in Section 1.5.

The action of \(G\) is called hamiltonian if there exists a momentum map

\[
\mu : M \to g^*
\]

from \(M\) to the dual \(g^*\) of \(g\), satisfying the following two properties:

(H1) For each \(a\) in \(g\), \(\mu^a := \langle \mu, a \rangle\) is a momentum of \(\hat{a}:

\[
d\mu^a = -i_{\hat{a}} \omega,
\]

and

(H2) \(\mu\) is \(G\)-equivariant:

\[
\mu(\gamma \cdot x) = \text{Ad}^*_\gamma (\mu(x)),
\]

for any \(x\) in \(M\) and any \(\gamma\) in \(G\).

In (6.3.6), \(\gamma \cdot x\) stands for the given action of \(G\) on \(M\) and \(\text{Ad}^*_\gamma\) denotes the coadjoint action of \(\gamma\) on \(g^*\), defined by \(\langle \text{Ad}^*_\gamma \zeta, a \rangle := \langle \zeta, \text{Ad}_\gamma^{-1} a \rangle\), for any \(\zeta\) in \(g^*\) and any \(a\) in \(g\).

**Proposition 6.3.1.** If \(G\) is connected, the equivariance condition (6.3.6) can be equivalently formulated as

\[
\mu^{[a, b]} = -\{\mu^a, \mu^b\},
\]

for any \(a, b\) in \(g\).

**Proof.** The condition (6.3.6) can be rewritten as

\[
\mu^{\text{Ad}^*_\gamma a}(\gamma \cdot x) = \mu^a(x),
\]

for any \(\gamma\) in \(G\), any \(a\) in \(g\) and any \(x\) in \(M\). By substituting \(\gamma_t := \exp(tb)\) and by derivating in \(t\) at \(t = 0\), we get \(\mu^{[b, a]}(x) + d\mu^a_{\gamma x}(b) = 0\), hence, by using (6.3.5),

\[
\mu^{[a, b]}(x) = -\omega_x(\hat{a}, \hat{b}),
\]
for any $a, b$ in $\mathfrak{g}$ and any $x$ in $M$. By \eqref{(1.2.10)}, $\omega(\dot{a}, \dot{b})$ is the Poisson bracket $\{\mu^a, \mu^b\}$. We then get \eqref{(6.3.7)}. Conversely, \eqref{(6.3.9)}-\eqref{(6.3.7)} implies \eqref{(6.3.6)}-\eqref{(6.3.8)} if $G$ is connected.

**Remark 6.3.1.** The above proposition can be rephrased as follows:

*If $G$ is connected, a momentum map for the action of $G$ on $M$ is the assignment of a momentum $\mu^a$ for each vector field $\dot{a}$ in such a way that the map $a \mapsto \mu^a$ be a Lie algebra anti-homomorphism from $\mathfrak{g}$ to $C^\infty(M, \mathbb{R})$, equipped with the Poisson bracket.* Note that the minus sign in the rhs of \eqref{(6.3.7)} is due to the fact that the action of $G$ on $M$ has been chosen a left action, which is also the reason why a minus sign appears in the rhs of \eqref{(6.3.2)}. If we substitute a right action by putting $R_\gamma(x) := \gamma^{-1} \cdot x$, the minus sign disappears in \eqref{(6.3.2)} as well as in \eqref{(6.3.7)} and the induced map from $\mathfrak{g}$ to $C^\infty(M, \mathbb{R})$ then becomes a Lie algebra homomorphism.

**Remark 6.3.2.** (i) A momentum map for a hamiltonian action of $G$ on $M$ is uniquely defined up to the addition of a $\text{Ad}^*$-invariant element of $\mathfrak{g}^*$, i.e. an element $\zeta$ of $\mathfrak{g}^*$ such that $\langle \zeta, a \rangle = 0$, for any $a$ in the derived Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. If $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, in particular if $\mathfrak{g}$ is semi-simple, the momentum map for a hamiltonian action is then uniquely defined.

(ii) If $(M, \omega)$ is compact and $H^1_{dR}(M, \mathbb{R}) = \{0\}$, any symplectic action of a connected Lie group $G$ on $M$ is hamiltonian. Indeed, for any $a$ in $\mathfrak{g}$, $\dot{a}$ is certainly hamiltonian, as $H^1_{dR}(M, \mathbb{R}) = \{0\}$, and if we define $\mu^a$ as the momentum of $\dot{a}$ normalized $\int_M \mu^a \omega^m = 0$, then \eqref{(6.3.7)} is certainly satisfied.

(iii) If $\mathfrak{g}$ is semi simple, its Killing form is non-degenerate and thus identifies $\mathfrak{g}$ with $\mathfrak{g}^*$ in a $G$-equivariant way. The momentum map — if any — can then be viewed as an equivariant map from $M$ to $\mathfrak{g}$.

We now apply the above general considerations to the action of the connected isometry group $K(\mathbb{P}^m, g_{FS,c}) = SU(m + 1)/\mathfrak{su}(m + 1)$ on $\mathbb{P}^m$, whose action preserves the full Kähler structure, in particular the symplectic form $\omega_{FS,c}$. Since $M = \mathbb{P}^m$ is compact and simply-connected we know that a momentum map exists, and since the Lie algebra $\mathfrak{g} = \mathfrak{su}(m + 1)$ is semi-simple, this momentum map is uniquely defined. Moreover, since the action of $G$ is transitive, the momentum map is an immersion — direct consequence of \eqref{(6.3.5)} — and its image is a unique coadjoint orbit of $SU(m + 1)/\mathfrak{su}(m + 1)$ in $\mathfrak{su}(m + 1)$. Since $\mathbb{P}^m$ is compact and simply-connected, the momentum map $\mu_{FS}$ then identifies $\mathbb{P}^m$ with a coadjoint orbit of $SU(m + 1)/\mathfrak{su}(m + 1)$ in $\mathfrak{su}(m + 1)$.

The explicit form of the momentum map $\mu_{FS,c} : \mathbb{P}^m \to \mathfrak{su}(m + 1)^*$ is easy to determine. From \eqref{(6.1.5)} we infer that for any $a$ in the Lie algebra $\mathfrak{su}(m + 1)$ the induced vector field $\dot{a}$ on $\mathbb{P}^m$ is given by

\begin{equation}
\dot{a}_x(u) = a(u) - (a(u), u) u, \tag{(6.3.10)}
\end{equation}

for any unit generator $u$ of the complex line $x$ (we here identify $\mathbb{C}^{m+1}/x$ with the (hermitian) orthogonal complement of $x$ in $\mathbb{C}^{m+1}$). From \eqref{(6.2.1)} and \eqref{(6.3.5)} we easily deduce the following expression of $\mu_{FS,c}^a = \langle \mu, a \rangle$:

\begin{equation}
\mu_{FS,c}^a(x) = -\frac{2}{c} i (a(u), u), \tag{(6.3.11)}
\end{equation}
for any \( a \) in \( \mathfrak{su}(m+1) \) and for any \( x \) in \( \mathbb{P}^m \) (here and henceforth, \( u \) stands for any unit generator of the complex line \( x \)). Note that

\[
(6.3.12) \quad \int_{\mathbb{P}^m} \mu_{FS,c}^a(x) \psi_{FS,c} = 0
\]

for any \( a \) in \( \mathfrak{su}(m+1) \), as required. This follows from the fact that the elements of \( \mathfrak{su}(m+1) \) are trace-free and from the general following expression for the trace of any endomorphism, \( A \), of \( \mathbb{C}^{m+1} \):

\[
(6.3.13) \quad \text{tr} A = \frac{m+1}{V(\mathbb{S}^{2m+1})} \int_{\mathbb{S}^{2m+1}} (Au, u) \psi_{\mathbb{S}^{2m+1}} = \frac{m+1}{V(\mathbb{P}^m)} \int_{\mathbb{P}^m} (Au, u) \psi_{FS}.
\]

Here, \( V(\mathbb{S}^{2m+1}) \) denotes the volume of the unit sphere \( \mathbb{S}^{2m+1} \) and \( V(\mathbb{P}^m) \) the volume of the standard projective space for the Fubini-Study of holomorphic sectional curvature \( c = 4 \), with \( V(\mathbb{S}^{2m+1}) = 2\pi V(\mathbb{P}^m) = \frac{2^{m+1}}{m!} \), cf. Remark 6.2.3.

By Remark 6.3.2, \( \mu_{FS,c} \) can be viewed as a map from \( \mathbb{P}^m \) to the Lie algebra \( \mathfrak{su}(m+1) \) itself, via the Killing form \( K^{\mathfrak{su}(m+1)} \) of \( \mathfrak{su}(m+1) \). The latter is easily checked to be given by

\[
(6.3.14) \quad K^{\mathfrak{su}(m+1)}(a, b) = 2(m+1)\text{tr}(a \circ b).
\]

As the Killing form of any semi-simple compact Lie group, it is negative definite, and is here conveniently replaced by the positive definite renormalized Killing form \( \tilde{K}^{\mathfrak{su}(m+1)}(a, b) = -\text{tr}(a \circ b) \). By using \( \tilde{K}^{\mathfrak{su}(m+1)} \), the momentum map \( \mu_{FS,c} \) becomes a map from \( \mathbb{P}^m \) to \( \mathfrak{su}(m+1) \) and has then the following expression:

\[
(6.3.15) \quad \mu_{FS,c}(x) = \frac{2i}{c} \left( \Pi_x - \frac{1}{m+1} I \right),
\]

where \( I \) denotes the identity of \( \mathbb{C}^{m+1} \) and \( \Pi_x \) stands for the hermitian orthogonal projector from \( \mathbb{C}^{m+1} \) to the complex line \( x \). In terms of matrices, we thus have

\[
(6.3.16) \quad (\mu_{FS,c}(x))_{ij} = \frac{2i}{c} \left( \sum_{k=0}^m u_k \bar{u}_j \frac{1}{\sqrt{m+1}} \delta_{ij} \right).
\]

for any \( i, j = 0, \ldots, m \), for any generator \( u = (u_0, u_1, \ldots, u_N) \) of the complex line \( x \).

### 6.4. The blowing up process

Let \( \mathbb{P}(V) \) be a complex projective space of (complex) dimension \((m - 1)\), \( m > 1 \); \( V \) is then a complex vector space of dimension \( m \). As a \( m \)-dimensional complex manifold, the (total space of) the tautological line bundle \( \Lambda = \mathcal{O}(-1) \) of \( \mathbb{P}(V) \) can be viewed as a submanifold of the product \( V \times \mathbb{P}(V) \), namely the set of pairs \((u, x)\) in \( V \times \mathbb{P}(V) \) such that \( u \) belongs to \( x \). Then, the second projection makes \( \Lambda \) into a holomorphic line bundle over \( \mathbb{P}(V) \), as we initially defined it, for which the zero section, say \( \Sigma_0 \), is the subset of pairs \((0, x)\) and is then isomorphic to \( \mathbb{P}(V) \) itself. Denote by \( p \) the map from \( \Lambda \) to \( V \) determined by the first projection: then, \( p^{-1}(0) = \Sigma_0 \), whereas \( p_{|\Lambda \setminus \Sigma_0} \) is an isomorphism from \( \Lambda \setminus \Sigma_0 \) to \( V \setminus \{0\} \). In other words, \( p \) realizes \( \Lambda \) as the blow-up of \( V \) at the origin \( 0 \), which consists in replacing the point \( 0 \) of \( V \) by the set of tangent complex directions at \( 0 \).
In terms of the natural (linear) holomorphic coordinates, say \( u_0, \ldots, u_{m-1} \), of \( V \) determined by a basis \( \{ e_0, \ldots, e_{m-1} \} \) of \( V \), the map \( p \) has the following expression. Denote by \( V_j \) the open set of \( V \) defined by the condition \( u_j \neq 0 \); then, \( V_j / \mathbb{C}^* \) is the affine subspace \( \mathbb{P}^\ast_j(V) \) of \( \mathbb{P}(V) \), cf. Section 6.1.

For simplicity, consider the case when \( j = 0 \); then, \( \mathbb{P}^\ast_0(V) \) is identified with \( \mathbb{C}^{m-1} \), by setting \( z_j = \frac{u_j}{u_0} \), \( j = 1, \ldots, m-1 \), and the map 
\[ \sigma_0 : (z_1, \ldots, z_{m-1}) \mapsto (1, z_1, \ldots, z_{m-1}) \]
\[ \text{is then a holomorphic section, without zero, of } \Lambda, \text{ over } \mathbb{P}^\ast_0(V) = \mathbb{C}^m \] (cf. Section 6.1). On the open subset \( \pi^{-1}(\mathbb{P}^\ast_0(V)) \) of \( \Lambda \) we then get a system of holomorphic coordinates, say 
\[ (\lambda, z_1, \ldots, z_{m-1}) \] by identifying \( (\lambda, z_1, \ldots, z_{m-1}) \) with \( \lambda \sigma((z_1, \ldots, z_{m-1})) \).

On \( \pi^{-1}(\mathbb{P}^\ast_0(V)) \), the map \( p \) has then the following analytical expression:

\[ p : (\lambda, z_1, \ldots, z_{m-1}) \mapsto (u_0 = \lambda, u_1 = \lambda z_1, \ldots, u_{m-1} = \lambda z_{m-1}) \]

The operation of blowing up can then be performed for any \( m \)-dimensional complex manifold \( M, m \geq 1 \), and for any point \( x_0 \) of \( M \). The procedure goes as follows: (i) choose a holomorphic chart domain \( U \) around \( x_0 \), in such a way that \( U \) be identified with a ball \( B \) in \( V \) centered as 0; (ii) replace \( U \) by \( \Lambda_U := p^{-1}(B) \subset \Lambda; \) more precisely, \( U \setminus \{ x_0 \} \) appears as an open set in \( \Lambda_U \) as well as in \( M \setminus \{ x_0 \} \) and we glue together \( \Lambda_U \) and \( M \setminus \{ x_0 \} \) along \( U \setminus \{ x_0 \} \), cf. e. g. [123] for details.

The resulting object, call it \( \tilde{M}_{x_0} \) or simply \( \tilde{M} \) if \( x_0 \) is understood, is then described by

\[ \tilde{M}_{x_0} = (M \setminus \{ x_0 \}) \bigcup_{U \setminus \{ x_0 \} = p^{-1}(B) \setminus \Sigma_0} p^{-1}(B) \]

meaning that \( \tilde{M}_{x_0} \) is obtained from the disjoint union \( M \setminus \{ x_0 \} \bigcup p^{-1}(B) \) by gluing these two pieces via the common identification of \( U \setminus \{ x_0 \} \subset M \setminus \{ x_0 \} \) and \( p^{-1}(B) \setminus \Sigma_0 \) with \( B \setminus \{ 0 \} \). This can be easily made into a \( m \)-dimensional complex manifold and comes equipped with a natural holomorphic map, say \( p \) again, from \( \tilde{M}_{x_0} \) to \( M \) defined as follows: (i) if \( y \) belongs to \( M \setminus \{ x_0 \} \), then \( p(y) = y \); (ii) if \( y \) belongs to \( \Sigma_0 \) in \( p^{-1}(B) \), then \( p(y) = x_0 \). This map is then an isomorphism from \( \tilde{M} \setminus p^{-1}(x_0) \) onto \( M \setminus \{ x_0 \} \), whereas the exceptional fiber — or exceptional divisor — \( \tilde{x}_0 := p^{-1}(x_0) \) is isomorphic to the complex projective space \( \mathbb{P}(T_{x_0}M) \).

---

1 For any (smooth) complex manifold \( M \), the group of divisors, \( \mathcal{D} \) of \( M \) is the free abelian group whose generators are the connected, irreducible subvarieties of \( M \) of codimension 1. A non-zero element of \( \mathcal{D} \) is then a formal sum \( D = \sum_{i=1}^N m_i D_i \), where each \( D_i \) is a connected, irreducible, complex analytic subvariety of \( M \) and the coefficients \( m_i \) are integers, which we can assume non-zero.

For any holomorphic line bundle, \( L \), over \( M \), each meromorphic section, \( s \), of \( L \) determines a divisor, where each \( D_i \) is a zero or a pole of \( s \) with multiplicity \( m_i \), positive if \( D_i \) is a zero, negative if \( D_i \) is a pole. Conversely, any divisor \( D \) is the divisor of a meromorphic section \( s^D \) of a holomorphic line bundle \( L^D \) over \( M \), uniquely defined up to isomorphism. If \( D \) is a smooth, connected submanifold of codimension 1, \( L_D \) is trivial outside some open neighbourhood of \( D \) in \( M \) and its restriction to \( D \) is isomorphic to the normal vector bundle \( \nu(D) \) of \( D \) in \( M \); moreover, the first Chern class of \( D \) is the Poincaré dual of the homology class of \( D \).

Let \( \mathcal{I}_D \) denote the defining sheaf of \( D \) in \( M \), i.e. the sheaf of germs of holomorphic functions which vanish on \( D \); we then have: \( \mathcal{I}_D = L_D \), meaning that \( \mathcal{I}_D \) is isomorphic to the sheaf of germs of holomorphic sections of the dual complex line bundle \( L_D^* \) (this...
The blowing up operation at $x_0$ depends upon the choice of a local holomorphic chart around $x_0$, but any two blow-up’s $\hat{M}$, $\hat{M}'$ of $M$ at the same point $x_0$ are isomorphic via complex isomorphisms which can be so chosen as to commute with the projections $p$, $p'$ and to induce the identity of $M$ outside an open neighbourhood of $x_0$. On the other hand, blow-up’s of a same complex manifold at two different points are in general non-isomorphic complex manifolds.

In general, for any (closed) complex submanifold, say $M_0$, of a complex manifold $M$, the normal bundle, $ν(M)$, of $M_0$ in $M$ is the holomorphic vector bundle over $M$ defined by $ν(M) = TM|Μ_0/TM$. By the very construction of $Μ_{x_0}$, the exceptional divisor $\hat{x}_0$ admits an open neighbourhood in $Μ_{x_0}$ isomorphic to $p^{-1}(B) \subset Λ$. The normal bundle of $\hat{x}_0$ in $\hat{Μ}_{x_0}$ is then isomorphic to $Ω = O(−1)$.

**Proposition 6.4.1.** For any compact complex manifold $M$ and any point $x_0$ on $M$, denote by $\hat{M}$ the blow-up of $M$ at $x_0$. Then, $H(M)$ is naturally realized as the subgroup of elements of $H(M)$ which fix $x_0$.

**Proof.** We first show that any (real) holomorphic vector field, $\hat{X}$, on $\hat{M}$ descends to a (real) holomorphic vector field, $X$, on $M$, with $X(x_0) = 0$. Indeed, since $p$ is a biholomorphism from $\hat{M} \setminus \hat{x}_0$ to $M \setminus \{x_0\}$, $\hat{X}$ certainly descends to a holomorphic vector field on $M \setminus \{x_0\}$. On the other hand, the restriction of $\hat{X}$ on the exceptional divisor $\hat{x}_0$ determines a holomorphic section of the normal bundle $N = T\hat{Μ}_{x_0}/T\hat{x}_0$, which, as we already observe, is isomorphic to $O(−1)$, hence has no non trivial global holomorphic section (cf. Prop 6.1.1); the restriction of $\hat{X}$ is then tangent to $\hat{x}_0$. It follows that $\hat{X}$ descends to a continuous vector field, $X$, on $M$, which is holomorphic outside $x_0$ and equal to 0 at $x_0$. By Hartogs’ theorem, $X$ is holomorphic everywhere on $M$, hence belongs to the Lie subalgebra, $h_{x_0}(M)$, of elements of $h(M)$ which vanish at $x_0$. Moreover, the resulting map from $h(M)$ to $h_{x_0}(M)$ is a Lie algebra morphism. This map is clearly injective and we now show that its is surjective, hence a Lie algebra isomorphism. Indeed, any element $X$ of $h_{x_0}$ lifts to a (real) holomorphic vector field, say $\hat{X}$, on $\hat{M} \setminus \hat{x}_0$, via the isomorphism $p$ from $\hat{M} \setminus \hat{x}_0$ to $M \setminus x_0$; we then complete $\hat{X}$ into a (real) vector field, still denoted by $\hat{X}$, on $\hat{M}$ by defining $\hat{X}$ on $\hat{x}_0$ as the (real) holomorphic vector field determined by the complex derivative of $X$ at $x_0$ viewed as an endomorphism of $T_{x_0}M$, cf. Remark 6.1.3 (since $X(x_0) = 0$, the complex derivative of $X$ at $x_0$, say $a$, is well defined; for example, $a = ∇X$ for any C-linear connection on $TM$). We now have to check that $\hat{X}$ is well defined as an element of $h(\hat{M})$. Since $X$ is holomorphic on $M \setminus x_0$ we only have to worry about the behaviour of $\hat{X}$ near points of the exceptional divisor $\hat{x}_0$ and since, $\hat{x}_0$ has a neighbourhood in $\hat{M}$ isomorphic to a neighbourhood of $Σ_0$ in $Λ$, we can use the local expression of $p$ given by (6.4.1), where $u_0,\ldots,u_m$ are the chosen holomorphic coordinates used to perform the blow-up. Relatively to these coordinates, $X$ — which will be here conveniently regarded as a holomorphic (complex) vector field of type holds, more generally, for any effective divisors, i.e. divisors $D = \sum_{j=1}^N m_j D_j$ where all $m_j$ positive). cf. e.g. [100, Chapter IV] for details.
(1, 0) — is written as: \( X = \sum_{j=1}^{m-1} X_j \partial/\partial u_j \), where the \( X_j \) are holomorphic functions of the \( u_0, \ldots, u_{m-1} \); in terms of the coordinates \( \lambda, z_1, \ldots, z_m \), \( X \) has then the following expression:

\[
X = X_0 \partial/\partial \lambda + \sum_{j=1}^{m-1} \left( \frac{X_j}{\lambda} - \frac{z_j X_0}{\lambda} \right) \partial/\partial z_j.
\]

This expression only makes sense a priori on the open set where \( \lambda \neq 0 \). On the other hand, near the origin, \( X_0 \) and the \( X_j \), \( j = 1, \ldots, m - 1 \), can be written as \( X_0 = a_0 u_0 + \sum_{k=1}^{m-1} a_{0k} u_k \), \( X_j = a_{0j} u_0 + \sum_{k=1}^{m-1} a_{jk} u_k \), mod terms of higher orders in the \( u_0, \ldots, u_{m-1} \), where \( a \) is the matrix of the complex derivative of \( X \) at 0 in terms of the coordinates \( u_0, \ldots, u_{m-1} \). By plugging into (6.4.3), we thus get:

\[
X = X_0 \partial/\partial \lambda + \sum_{j=1}^{m-1} \left( \sum_{k=1}^{m-1} (a_{0j} + a_{jk} z_k - a_{00} z_j - a_{0k} z_j z_k) \right) \partial/\partial z_j
\]

mod terms which are of order at least 1 in \( \lambda \). This expression is now well defined, even for \( \lambda = 0 \), and, for \( \lambda = 0 \), reduces to (6.1.6). This show that \( X \) is well defined on the whole of \( M \) as an element of \( h(M) \) and that the restriction of \( X \) to \( \tilde{x}_0 \) coincides, as was claimed, with the holomorphic vector field on \( \mathbb{P}(T_{x_0} M) \) by the complex derivative of \( X \) at \( x_0 \).

Let \( \tilde{\gamma} \) be an element of \( H(\tilde{M}) \) and assume that \( \tilde{\gamma} \) belongs to the image of the exponential map from \( h(\tilde{M}) \) to \( H(\tilde{M}) \), i.e. is the value at \( t = 1 \) of the flow \( \Phi_t^{\tilde{X}} \) of some element \( \tilde{X} \) of \( h(\tilde{M}) \); then, \( \tilde{X} \) descends to a well defined element \( X \) of \( h_{x_0} \), whose flow, \( \Phi_t^X \), belongs to \( H(M, J) \) and fixes \( x_0 \) for each \( t \); moreover, \( \Phi_t^X = \Phi_t^{\tilde{X}} \circ \tilde{p} \) for each \( t \). It follows that \( \tilde{\gamma} \) globally preserves the exceptional divisor \( \tilde{x}_0 \), hence descends to a homeomorphism of \( M \), say \( \gamma \), which fixes \( x_0 \) and is holomorphic on \( M \setminus \{x_0\} \); by Hartogs’ theorem again, \( \gamma \) is holomorphic everywhere, hence belongs to \( H_{x_0}(M) \), if \( H_{x_0}(M) \) denotes the (closed) Lie subgroup of elements of \( H(M, J) \) which fix \( x_0 \). In general, not every element of \( H(M) \) belongs to image the exponential map, but any \( \tilde{\gamma} \) in \( H(M) \) is the product of a finite number of such elements: it follows that any \( \gamma \) in \( H(\tilde{M}) \) (globally) preserves \( \tilde{x}_0 \), hence descends to an element, \( \gamma \), of \( H(M, J) \) which fixes \( x_0 \). The resulting homomorphism, say \( \tilde{p} \), from \( H(\tilde{M}) \) to \( H_{x_0}(M) \), covers the above Lie algebra isomorphism \( p \) from \( h(\tilde{M}) \) to \( h_{x_0}(M) \); it is then a covering map; on the other hand, \( \tilde{p} \) is clearly injective (if \( \tilde{\gamma} \) in \( H(\tilde{M}) \) is the identity on \( \tilde{M} \setminus \tilde{x}_0 \), it certainly restricts to the identity on \( \tilde{x}_0 \); \( \tilde{p} \) is then an Lie group isomorphism. Observe that \( \tilde{\gamma} \) can be recovered from \( \gamma \) in the following manner: (i) the action of \( \tilde{\gamma} \) on \( \tilde{M} \setminus \tilde{x}_0 \) coincides with the action of \( \gamma \) on \( M \setminus \{x_0\} \); (ii) the action of \( \tilde{\gamma} \) on the exceptional divisor \( \tilde{x}_0 = \mathbb{P}(T_{x_0} M) \) coincides with the induced linear action of \( \gamma \) on \( T_{x_0} M \).

**Proposition 6.4.2.** Denote by \( L_{\tilde{x}_0} \) the holomorphic line bundle on \( \tilde{M} \) determined by the exceptional divisor \( \tilde{x}_0 \), cf. footnote of page 154. Then, the canonical line bundles, \( K(\tilde{M}) \) and \( K(M) \), of \( \tilde{M} \) and \( M \) are related by:

\[
K(\tilde{M}) = p^* K(M) \otimes L_{\tilde{x}_0}^{m-1}.
\]
In particular, the first Chern classes of \( \hat{M} \) and \( M \) are related by

\[
(6.4.6) \quad c_1(\hat{M}) = p^*c_1(M) - (m - 1)c_1(L_{\hat{x}_0}).
\]

**Proof.** Outside the exceptional divisor \( \hat{x}_0 \), we certainly have \( K(\hat{M}) = p^*K(M) \), whereas a neighbourhood of \( \hat{x}_0 \) in \( \hat{M} \) is covered by \( m \) holomorphic charts, \( \hat{U}^{(0)}, \ldots, \hat{U}^{(m-1)} \), with holomorphic coordinates \((\lambda = \lambda^{(0)}, z_1 = z_1^{(0)} \), \ldots, \( z_{m-1} = z_{m-1}^{(0)} \)), in such a way that \( p \) be locally expressed in each \( \hat{U}^{(0)} \) as in (6.4.1), and on each \( \hat{U}^{(j)} \), \( j > 0 \), by \( p : (\lambda^{(j)}, z_1^{(j)} \), \ldots, \( z_{m-1}^{(j)} \) \( \mapsto \) \((\lambda^{(j)}z_1^{(j)} \), \ldots, \( \lambda^{(j)}z_{m-1}^{(j)} \), \( \lambda^{(j)}z_{1-j}^{(j)} \), \ldots, \( \lambda^{(j)}z_{m-1}^{(j)} \)).

On each \( \hat{U}^{(j)} \), \( \lambda^{(j)} \) is an equation of the exceptional divisor \( \hat{x}_0 \), meaning that, if \( \mathcal{L}_{\hat{x}_0} \) denotes the defining sheaf of \( \hat{x}_0 \) — whic, we recall, is isomorphic to \( L_{\hat{x}_0} \), cf. footnote of page 154 — \( \mathcal{L}_{\hat{x}_0}(\hat{U}^{(j)}) \) is freely generated by \( \lambda^{(j)} \) over \( \mathcal{O}(\hat{U}^{(j)}) \) (the space of holomorphic functions defined on \( \hat{U}^{(j)} \)). From (6.4.1), we get \( p^*(du_0 \wedge du_1 \wedge \ldots \wedge du_{m-1}) = \lambda^{m-1}d\lambda \wedge dz - 1 \wedge \ldots \wedge dz_{m-1} \) on \( \hat{U}^{(0)} \), and we get similar expression on the other open sets \( \hat{U}^{(j)} \). We thus get \( p^*K(M) = K(\hat{M}) \otimes (L_{\hat{x}_0})^{m-1} \), which readily implies (6.4.5), as well as (6.4.6) (recall that, for any complex manifold \( M \), \( c_1(M) \) is defined by as the first Chern class of the anti-canonical line bundle, i.e. \( c_1(M) = c_1(K_M^{-1}) = -c_1(K_M) \)).

## 6.5. Blow-up of projective spaces and Hirzebruch surfaces

The blow-up at some point \( x_0 \) of a complex projective space \( \mathbb{P}(V) \), with \( V \cong \mathbb{C}^{m+1} \) and \( m \geq 1 \), can be described in the following way (since \( H(\mathbb{P}(V)) = PG(V) \) acts transitively on \( \mathbb{P}(V) \), the blow-up, in this case, is essentially independent of the choice of \( x_0 \)). Denote by \( Q_{x_0} \) the space of projective lines in \( \mathbb{P}(V) \) which pass through \( x_0 \) \(^2\) and define \( \hat{\mathbb{P}}(V)_{x_0} \) as the set of pairs \((x, D)\) in \( \mathbb{P}(V) \times Q_{x_0} \) such that \( x \) belongs to \( D \):

\[
(6.5.1) \quad \hat{\mathbb{P}}(V)_{x_0} = \{(x, D) \in \mathbb{P}(V) \times Q_{x_0} \mid x \in D\}.
\]

Denote by \( p \) the projection from \( \hat{\mathbb{P}}(V)_{x_0} \) to \( \mathbb{P}(V) \) determined by the obvious projection from \( Q_{x_0} \times \mathbb{P}(V) \) to \( \mathbb{P}(V) \). Then, \( p \) is clearly an isomorphism outside \( p^{-1}(x_0) \), whereas \( p^{-1}(x_0) \) is the whole of \( Q_{x_0} \), which is naturally identified with \( \mathbb{P}(T_{x_0} \mathbb{P}(V)) \).

In order to get a more concrete grasp on \( \hat{\mathbb{P}}(V)_{x_0} \), it is convenient to choose a complex vector subspace, \( V_0 \) say, of \( V \) such that \( V = x_0 \oplus V_0 \). \( \mathbb{P}(V_0) \) is then a projective hyperplane of \( \mathbb{P}(V) \) which does not meet \( x_0 \). It follows that each projective line \( D \) of \( \mathbb{P}(V) \) in \( Q_{x_0} \) meets \( \mathbb{P}(V_0) \) in a unique point, so that \( Q_{x_0} \) is naturally identified with \( \mathbb{P}(V_0) \). Denote by \( \Lambda^{V_0} \) the tautological line bundle of \( \mathbb{P}(V_0) \).

\(^2\)A projective line in \( \mathbb{P}(V) \) is a complex projective line \( D = \mathbb{P}(P) \), where \( P \) is any complex 2-plane, i.e. any complex vector subspace of dimension 2, of \( V \); the points of \( D \) are then the complex lines of \( V \) which are contained in \( P \); \( Q_{x_0} \) is then the space of complex vector 2-planes of \( V \) which contain the complex line \( x_0 \).
Proposition 6.5.1. For any choice of a complement $V_0$ of $x_0$ in $V$, we have a natural identification of complex manifolds:

$$\mathbb{P}(V)_{x_0} = \mathbb{P}(1_{x_0} \oplus \Lambda^{V_0}).$$

(6.5.2)

Here $1_{x_0}$ stands for the trivial line bundle $\mathbb{P}(V_0) \times x_0$ and $\mathbb{P}(1_{x_0} \oplus \Lambda^{V_0})$ denotes the holomorphic projective line bundle over $\mathbb{P}(V_0)$ whose fiber at $y$ is the projective line $\mathbb{P}(x_0 \oplus y)$, when $y$ is viewed as a complex line of $V_0$.

Proof. The identification (6.5.2) is essentially tautological. To any pair $(x, D)$ in $\mathbb{P}(V)_{x_0}$ we first associate the complex line $y := D \cap \mathbb{P}(V_0)$ of $V_0$, viewed as a point of $\mathbb{P}(V_0)$. Then, $D = \mathbb{P}(x_0 \oplus y)$ and $x$ can then be regarded as a point of the fiber at $y$ of $\mathbb{P}(1_{x_0} \oplus \Lambda^{V_0})$. Conversely, any point $x$ of $\mathbb{P}(1_{x_0} \oplus \Lambda^{V_0})$ over $y$ is a complex line in the 2-plane $x_0 \oplus y$ of $V$, hence a point of $\mathbb{P}(V)$. The corresponding element of $\mathbb{P}(V)_{x_0}$ is then the pair $(x, D)$, where $D = \mathbb{P}(x_0 \oplus y)$ is viewed as a projective line in $\mathbb{P}(V)$.

It is convenient to identify the chosen complex line $x_0$ with $\mathbb{C}$, by choosing a generator of $x_0$, and $\mathbb{P}(1_{x_0} \oplus \Lambda^{V_0})$ is then identified with the projective line bundle $\mathbb{P}(1 \oplus \Lambda^{V_0})$, where 1 stands for the trivial line bundle $\mathbb{P}(V) \times \mathbb{C}$.

The projective line bundle $\mathbb{P}(1_{x_0} \oplus \Lambda^{V_0})$ is now defined independently of the projective space $\mathbb{P}(V)$ and Proposition 6.5.1 can then be reformulated as follows: For any complex projective space $\mathbb{P}(V_0)$, we have a canonical identification

$$\mathbb{P}(1 \oplus \Lambda^{V_0}) = \mathbb{P}(\mathbb{C} \oplus V_0),$$

(6.5.3)

where the rhs denotes the blow-up of the complex projective space $\mathbb{P}(\mathbb{C} \oplus V_0)$ at the point $\mathbb{C}$ as defined above, i.e. as the incidence submanifold of $\mathbb{P}(V_0) \times \mathbb{P}(\mathbb{C} \oplus V_0)$.

In the sequel, the natural projection from $\mathbb{P}(1 \oplus \Lambda^{V_0})$ to $\mathbb{P}(V_0)$ will be denoted $\pi$.

The tautological line bundle $\Lambda^{V_0}$ is naturally embedded as an open set — actually an open subbundle — of $\mathbb{P}(1 \oplus \Lambda^{V_0})$ by identifying any element $u$ of $\Lambda^{V_0}$, over some point $y$ of $\mathbb{P}(V_0)$, with the complex line, $x$, of the 2-plane $\mathbb{C} \oplus y$ generated by $(1, u)$. For any $y$ in $\mathbb{P}(V_0)$ we then have

$$\Lambda^{V_0}_y = \pi^{-1}(y) \setminus \{y\},$$

(6.5.4)

where, in the rhs, $y$ is viewed as the complex line $y$ in the 2-plane $\mathbb{C} \oplus y$, hence as a point in $\pi^{-1}(y)$.

The projective line bundle $\mathbb{P}(1 \oplus \Lambda^{V_0})$ thus appears as a compactification of $\Lambda^{V_0}$ obtained by adding a point at infinity to each fiber $\Lambda^{V_0}_y$, namely the complex line $y$ itself as a distinguished point of $\pi^{-1}(y)$. The reunion of the points at infinity is denoted by $\Sigma_\infty$; $\Sigma_\infty$ is then the image of the (holomorphic) infinity section $\sigma_\infty : y \mapsto y \subset \mathbb{C} \oplus y$. For simplicity, $\Sigma_\infty$ itself will be called the infinity section of $\mathbb{P}(1 \oplus \Lambda^{V_0})$.

Via the embedding of $\Lambda^{V_0}$ in $\mathbb{P}(1 \oplus \Lambda^{V_0})$, the zero section of $\Lambda^{V_0}$ can be viewed as a complex submanifold, $\Sigma_0$, of $\mathbb{P}(1 \oplus \Lambda^{V_0})$, namely the image of the (holomorphic) zero section $\sigma_0 : y \mapsto \mathbb{C} \subset \mathbb{C} \oplus y$ of $\mathbb{P}(1 \oplus \Lambda^{V_0})$. For simplicity again, $\Sigma_0$ itself will be called the zero section of $\mathbb{P}(1 \oplus \Lambda^{V_0})$. Both $\Sigma_0$ and $\Sigma_\infty$ are isomorphic to $\mathbb{P}(V_0)$ via the projection $\pi$. 


Via the isomorphism (6.5.3), the blow-up map \( p \), viewed as a map from \( \mathbb{P}(1 \oplus \Lambda^0) \) to \( \mathbb{P}(V) \), \( V = \mathbb{C} \oplus V_0 \), can be described in the following way: any \( x \) in \( \pi^{-1}(y) \) is defined as an element of \( \mathbb{P}(x_0 \oplus y) \), i.e. as a complex line of \( V \) contained in the complex 2-plane \( x_0 \oplus y \) of \( V \), and we then have \( p(x) = x \), where \( x \) is simply regarded as a complex line in \( V \), i.e. as a point of \( \mathbb{P}(V) \), with no more reference to the 2-plane \( x_0 \oplus y \) (\( p \) then appears as a forgetting map). It follows that \( p^{-1}(x_0) \) is the whole zero section, \( \Sigma_0 \), whereas for any \( x \neq x_0 \), \( p^{-1}(x) \) is simply \( x \) itself, viewed as an element of \( \pi^{-1}(y) \), if \( y \) denotes the intersection of \( \mathbb{P}(V_0) \) with the complex projective line in \( \mathbb{P}(V) \) determined by \( x \) and \( x_0 \). Observe that \( p \) maps \( \Sigma_\infty \) biholomorphically to \( \mathbb{P}(V_0) \), whereas \( \Sigma_0 \) is mapped to \( x_0 \).

**Remark 6.5.1.** In general, for any complex manifold \( M \) and any holomorphic line bundle, \( L \), over \( M \), the associated projective line bundle \( \pi : \mathbb{P}(1 \oplus L) \to M \) can similarly be viewed as a compactification of \( L \) obtained by adding an *infinity section* \( \Sigma_\infty \), image of the section \( \sigma_\infty : y \mapsto L_y \subset \mathbb{C} \oplus L_y \) of \( \pi \), whereas the *zero section*, \( \Sigma_0 \), is the image of the section \( \sigma_0 : y \mapsto \mathbb{C} \subset \mathbb{C} \oplus L_y \).

The *first Hirzebruch surface*, \( \mathbb{F}_1 \), is the (compact) complex surface \( \mathbb{P}(1 \oplus \Lambda) \), when \( \Lambda \) denotes the tautological line bundle of the complex projective line \( \mathbb{P}^1 \). In view of (6.5.3), \( \mathbb{F}_1 \) is isomorphic to the blow-up of the complex projective plane \( \mathbb{P}^2 = \mathbb{P}(\mathbb{C} \oplus \mathbb{C}^2) \) at \( \mathbb{C} \).

For any positive integer \( \ell > 1 \), the \( k \)-th Hirzebruch surface, \( \mathbb{F}_\ell \), is defined similarly, by substituting \( \Lambda^\ell \) to \( \Lambda \), i.e. \( \mathbb{F}_\ell = \mathbb{P}(1 \oplus \Lambda^\ell) \). It turns out that the zero section \( \Sigma_0 \) can again be blown down, but the resulting object is no longer a smooth complex manifold but a projective variety with orbifold singularities, namely the *weighted projective plane* of weight \((1, 1, \ell)\), cf. [71].

**Remark 6.5.2.** It readily follows from the above discussion that the first Hirzebruch surface \( \mathbb{F}_1 \) can be realized as the complex submanifold of the product \( \mathbb{P}^1 \times \mathbb{P}^2 \) defined by the equation

\[
\tag{6.5.5}
x_1 y_2 - x_2 y_1 = 0,
\]

for \( x = (x_0 : x_1 : x_2) \) in \( \mathbb{P}^2 \) and \( y = (y_1 : y_2) \) in \( \mathbb{P}^1 \). Here, \( \mathbb{P}^2 \) is understood as the standard complex projective plane \( \mathbb{P}(\mathbb{C}^3) \), where \( \mathbb{C}^3 \) comes equipped with its natural basis \( e_0, e_1, e_2 \), and \( \mathbb{P}^1 \) is then \( \mathbb{P}(\mathbb{C}^2) \), where \( \mathbb{C}^2 \) is understood as the complex 2-plane generated by \( e_1, e_2 \). The identification goes as follows: to any pair \( (x = (x_0 : x_1 : x_2), y = (y_1 : y_2)) \) satisfying (6.5.5), we associate the complex line in the complex 2-plane \( \mathbb{C} e_0 \oplus y \) — the fiber at \( h \) of the vector bundle \( 1 \oplus \Lambda \) over \( \mathbb{P}^1 \) — generated by \( x_0 e_0 + \lambda (y_1 e_1 + y_2 e_2) \), with \( \lambda = \frac{x_1}{y_1} = \frac{x_2}{y_2} \). In this identification, the projection to the second factor \( \mathbb{P}^1 \) determines the natural projection from \( \mathbb{F}_1 \) to \( \mathbb{P}^1 \) whereas the projection to the first factor \( \mathbb{P}^2 \) realizes \( \mathbb{F}_1 \) as the blow-up of \( \mathbb{P}^2 \) at the point \((1 : 0 : 0)\). Moreover, the zero section \( \Sigma_0 \) is identified with the set of pairs \((x = (0 : 0 : 1), y = (y_0 : y_1))\), whereas the infinity section \( \Sigma_\infty \) is identified with the set of pairs \((x = (0 : y_0 : y_2), y = (y_1 : y_2))\) for all \((y_0 : y_1)\) in \( \mathbb{P}^1 \).

More generally, any Hirzebruch surface can be realized as the set of solutions of the equation

\[
\tag{6.5.6}
x_1 y_2 - x_2 y_1 = 0,
\]
in \( \mathbb{P}^2 \times \mathbb{P}^1 \). With the above notation, we simply identify each pair \((x = (x_0 : x_1 : x_2), y = (y_1 : y_2))\) satisfying (6.5.6) with the element of the fiber at \(y\) of \(\mathbb{P}(1 \oplus \Lambda')\) over generated by \(x_0 e_0 + \lambda (y_1 e_1 + y_2 e_1)^{\otimes \ell}\), with \(\lambda = \frac{z_2}{\ell_1} = \frac{z_2}{\ell_2}\).

For any \(\ell\), the Hirzebruch surface \(\mathbb{F}_\ell\) can be realized as a blow-up by substituting the weighted projective plane \(\mathbb{P}^2_{\ell,1,1}\) to \(\mathbb{P}^2\). In general, for any set \((a_0, \ldots, a_m)\) of positive integers, with \(\gcd(a_0, \ldots, a_m) = 1\), the corresponding weighted (complex) projective space \(\mathbb{P}^{m+1}_{a_0, \ldots, a_m}\) is defined as the quotient of \(\mathbb{C}^{m+1} \setminus \{0\}\) by the weighted action of \(\mathbb{C}^*\) defined by

\[
(6.5.7) \quad \zeta \cdot (z_0, \ldots, z_m) = (\zeta^{a_0} z_0, \ldots, \zeta^{a_m} z_m),
\]

for any \(\zeta \in \mathbb{C}^*\) and any \((z_0, \ldots, z_m)\) in \(\mathbb{C}^{m+1}\). We here consider the case when \(m = 2\) and \((a_0, a_1, a_2) = (\ell, 1, 1)\). The resulting weighted projective plane is smooth except at the point \((1 : 0 : 0)\) where it has an orbifold singularity of type \(\mathbb{C}^2/\mathbb{Z}_\ell\), where \(\mathbb{Z}_\ell\) stands for the group of \(\ell\)-roots of 1. The weighted projective plane \(\mathbb{P}^2_{\ell,1,1}\) admits a canonical map to the standard complex projective space \(\mathbb{P}^2\) — viewed as the weighted projective space \(\mathbb{P}^2_{1,1,1}\) — determined by the map

\[
(6.5.8) \quad q_\ell : (z_0, z_1, z_2) \mapsto (x_0 = z_0, x_1 = z'_1, x_2 = z'_2)
\]

from \(\mathbb{C}^3\) to itself (this map is clearly equivariant for the appropriate \(\mathbb{C}^*\)-actions, of weight \((\ell, 1, 1)\) in the lhs, of weight \((1, 1, 1)\) in the rhs, hence determines a well-defined (holomorphic) map from \(\mathbb{P}^2_{\ell,1,1}\) to \(\mathbb{P}^2\)). We now show that \(\mathbb{F}_\ell\) can be realized in the product \(\mathbb{P}^2_{\ell,1,1} \times \mathbb{P}^1\) by the equation

\[
(6.5.9) \quad z_1 y_2 - z_2 y_1 = 0.
\]

Denote by \(\bar{\mathbb{F}}_\ell\) the subvariety of \(\mathbb{P}^2_{\ell,1,1} \times \mathbb{P}^1\) defined by (6.5.9) and consider the map, \(\bar{q}_\ell\), from \(\mathbb{P}^2_{\ell,1,1} \times \mathbb{P}^1\) to \(\mathbb{P}^2 \times \mathbb{P}^1\) determined by \(q_\ell\) on the first factor and the identity on the second factor. From (6.5.9) we deduce \(z'_1 y_2 - z'_2 y_1 = 0\), hence \(x_1 y'_2 - x_2 y'_1 = 0\): this shows that \(\bar{q}_\ell\) maps \(\bar{\mathbb{F}}_\ell\) onto \(\mathbb{F}_\ell\) in \(\mathbb{P}^2 \times \mathbb{P}^1\). Moreover, the restriction of \(\bar{q}_\ell\) to \(\bar{\mathbb{F}}_\ell\) is injective. Indeed, for any pair \((x = (x_0 : x_1 : x_2), y = (y_1 : y_2))\) in \(\mathbb{F}_\ell\), the elements of \(\bar{\mathbb{F}}_\ell\) which map to \((x, y)\) are the pairs \((z = (z_0 : z_1 : z_2), y = (y_1 : y_2))\) such that \(z_0 = x_0, z'_1 = x_1, z'_2 = x_2\) and \(z_1 y_2 = z_2 y_1\). Since \(y_1, y_2\) cannot be both zero, we may assume in the following argument that \(y_1 \neq 0\), so that \(z_2\) is determined by \(z_1\). The triple \((z_0, z_1, z_2)\) is then entirely determined by the choice of \(z_1\) as an \(\ell\)-th root of \(y_1\). Moreover, if \(z_1\) is replaced by \(\zeta z_1\), for \(\zeta\) any \(\ell\)-th root of 1, then the corresponding new triple is \((z_0, \zeta z_1, \zeta z_2) = (\zeta^\ell z_0, \zeta z_1, \zeta z_2)\), which is equal to \(\zeta \cdot (z_0, z_1, z_2)\), for the \(\mathbb{C}^*\)-action of weight \((\ell, 1, 1)\), hence determines the same element of \(\mathbb{P}^2_{\ell,1,1}\).

### 6.6. Fano complex surfaces

A compact complex manifold \((M, J)\) whose first Chern class \(c_1(M, J)\) is positive, i.e. belongs to the Kähler cone, is called a Fano manifold. By its very definition any Fano manifold admits Kähler metrics polarized by
its anti-canonical line bundle \( K_{M}^{-1} \), cf. Section 1.19. Any compact Kähler-Einstein manifold with positive (constant) scalar curvature is a Fano manifold but not any Fano manifold admits a Kähler-Einstein metric, cf. Section 1.20.

Fano complex surfaces are well understood [196], [101]: these are \( \mathbb{P}^2 \), \( \mathbb{P}^1 \times \mathbb{P}^1 \) and the blow-ups of \( \mathbb{P}^2 \) at \( k \) distinct points \( p_1, \ldots, p_k \) in general position, with \( k \leq 8 \). Here in general position means that: (i) if \( k \geq 3 \), no three of them sit on a same projective line; (ii) if \( k \geq 6 \), no 6 of them sit on a same conic; (iii) if \( k = 8 \), there is no singular cubic passing through all of them and having one of them as a singular (double) point.

The complex projective plane \( \mathbb{P}^2 \) admits a Kähler-Einstein metric, the Fubini-Study metric, which is actually of constant holomorphic sectional curvature and is unique up to rescaling and the action of \( H(\mathbb{P}^2) \), cf. Section 6.2.

Similarly, the product \( \mathbb{P}^1 \times \mathbb{P}^1 \) admits a Kähler-Einstein metric, namely the product of two Fubini-Study metrics with the same value of the constant holomorphic sectional curvature. Again this metric is unique up to rescaling and the action of \( \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \). Note that the product of two Fubini-Study metrics with different (constant) holomorphic sectional curvatures is still of constant scalar curvature, but no longer Kähler-Einstein.

**Proposition 6.6.1.** The blow-up’s of \( \mathbb{P}^2 \) at one point or at any two distinct points are Fano but admit no Kähler-Einstein metric, not even Kähler metrics of constant scalar curvature.

**Proof.** This is a simple application of the non-existence criterion given by Lichnerowicz-Matsushima theorems 3.6.1, 3.6.2. For any \( k \), denote by \( M_k \) any blow-up of \( \mathbb{P}^2 \) at \( k \) distinct points \( p_1, \ldots, p_k \) in general position as above. By Proposition 6.4.1, \( H(M_k) \) is naturally identified with the subgroup of elements of \( H(\mathbb{P}^2) \) which preserve each point \( p_1, \ldots, p_k \). It remains to check that \( H(M_k) \) is not reductive whenever \( k = 1 \) or \( k = 2 \). Recall — cf. Proposition 6.1.2 — that \( H(\mathbb{P}^2) = PGL(3, \mathbb{C}) \). Since \( H(\mathbb{P}^2) \) acts transitively on \( \mathbb{P}^2 \) and on pairs of (distinct) points of \( \mathbb{P}^2 \), we can assume without loss of generality that \( M_1 \) is the blow-up of \( \mathbb{P}^2 \) at \( x_1 = (1 : 0 : 0) \) and that \( M_2 \) is the blow-up of \( \mathbb{P}^2 \) at \( x_1 = (1 : 0 : 0) \) and \( x_2 = (0 : 1 : 0) \). A typical element of \( H(M_1) \) can then be written as

\[
\gamma = \begin{pmatrix} 1 & u & v \\ 0 & a & b \\ 0 & c & d \end{pmatrix}
\]

where \( u, v \) stand for any two complex numbers and \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) for any element of \( GL(2, \mathbb{C}) \), whereas a typical element of \( H(M_2) \) is written as

\[
\gamma = \begin{pmatrix} 1 & 0 & v \\ 0 & a & b \\ 0 & 0 & d \end{pmatrix}
\]

where \( v \) stands for any complex number and \( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \) for element of the group \( T^+(2, \mathbb{C}) \) of invertible upper \( 2 \times 2 \) complex triangular matrix. In
other words, we have that

$$H(M_1) \equiv \mathfrak{gl}(2, \mathbb{C}) \ltimes \mathbb{C}^2,$$

the semi-direct product of the full complex linear group \( \mathfrak{gl}(2, \mathbb{C}) \) with \( \mathbb{C}^2 \) for the natural action of \( \mathfrak{gl}(2, \mathbb{C}) \) on \( \mathbb{C}^2 \), whereas

$$H(M_2) \equiv T^+(2, \mathbb{C}) \ltimes \mathbb{C},$$

semi-direct product of \( T^+(2, \mathbb{C}) \) and \( \mathbb{C} \) for the action of \( T^+(2, \mathbb{C}) \) on \( \mathbb{C} \) given by

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot v = d \cdot v.$$

At the level of Lie algebras, we then have

$$\mathfrak{h}(M_1) = \mathfrak{gl}(2, \mathbb{C}) \oplus \mathbb{C}^2,$$

with bracket given by \([([A, x], [B, y]) = ([A, B], Ax - By)]\), for any \( A, B \) in \( \mathfrak{gl}(2, \mathbb{C}) \), any \( x, y \) in \( \mathbb{C}^2 \), and

$$\mathfrak{h}(M_2) = t^+(2, \mathbb{C}) \oplus \mathbb{C},$$

with bracket similarly given by: \([([A, x], [B, y]) = ([A, B], Ay - Bx)]\), for any \( A, B \) in \( t^+(2, \mathbb{C}) \), any \( x, y \) in \( \mathbb{C} \). We easily check that the center of \( \mathfrak{h}(M_1) \) and the center of \( \mathfrak{h}(M_2) \) are both reduced to \( \{0\} \), whereas \( \mathfrak{h}(M_1) \) and \( \mathfrak{h}(M_2) \) are clearly not semi-simple: they are then both non-reductive. It then follows from Theorems 3.6.1, 3.6.2 that \( M_1 \) and \( M_2 \) admit no Kähler metric with constant scalar curvature, a fortiori no Einstein-Kähler metric.

Since \( H(\mathbb{P}^2) \) acts transitively on all triple of distinct points of \( \mathbb{P}^2 \), any \( M_3 \) is isomorphic to the blow-up of \( \mathbb{P}^2 \) at the points \( x_1 = (1 : 0 : 0), x_2 = (0 : 1 : 0) \) and \( x_3 = (0 : 0 : 1) \). A typical element of \( H(M_3) \) is then written as

$$\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$$

with \( a, b \) any two complex numbers such that \( ab \neq 0 \). We then have

$$H(M_3) = \mathbb{C}^* \times \mathbb{C}^*,$$

\footnote{Let \( G, H \) be any two Lie groups and suppose that \( G \) acts on \( H \) by group automorphisms. The semi-direct product, \( G \ltimes H \), of \( G \) by \( H \) relative to this action is then defined as the Lie group whose underlying manifold is the product \( G \times H \), with the multiplication law: \((g_1, h_1)(g_2, h_2) = (g_1g_2, (g_2^{-1} \cdot h_1)h_2)\), for any \( g_1, g_2 \) in \( G \) and any \( h_1, h_2 \) in \( H \); then, the maps \( g \mapsto (g, 1) \) and \( h \mapsto (1, h) \) realize \( G \) and \( H \) as subgroups of \( G \ltimes H \). \( H \) is normal in \( G \times H \), and the initial action of \( G \) on \( H \) simply becomes the natural action of \( G \) on \( H \) by inner automorphisms inside \( G \times H \).}

The semi-direct products considered in this section are of the type \( G \ltimes V \), where \( G \) is a Lie group and \( \rho : G \to \text{Aut}(V) \) is a linear representation of \( G \); each elements of \( G \ltimes V \) is a pair \((g, v)\) in \( G \times V \), and the product is defined by: \((g_1, v_1)(g_2, v_2) = (g_1g_2, \rho(g_2^{-1})(v_1) + v_2)\). Similarly, the elements of \( \text{Lie}(G \ltimes V) \) are pairs \((a, u)\) in \( \mathfrak{g} \times V \), and the bracket is then:

$$[[a_1, u_1], [a_2, u_2]] = ([a_1, a_2], dp(a_1)(u_2) - dp(a_2)(u_1)),$$

where \( dp : \mathfrak{g} \to \text{End}(V) \) denotes the induced representation of the Lie algebra \( \mathfrak{g} \) of \( G \), i.e. derivative at the origin of \( \rho \). In particular, it is easy to check that the Killing form, say \( B^{G \ltimes \rho V} \), is then given by

$$B^{G \ltimes \rho V}((a_1, u_1), (a_2, u_2)) = B^G(a_1, a_2) + \text{tr}(dp(a_1) \circ dp(a_2)).$$

In particular, \( V \), as an (abelian) Lie subalgebra of \( \text{Lie}(G \ltimes \rho V) \), is always contained in the kernel of \( B^{G \ltimes \rho V} \) and \( \text{Lie}(G \ltimes \rho V) \) is then never semisimple, unless \( \rho \) is a trivial representation and \( \mathfrak{g} \) is semisimple.
which is abelian, hence reductive. It is easy to check that $\text{Aut}_0(M_k) = \{1\}$ if $k \geq 4$. It follows that $\text{Aut}_0(M_k)$ is reductive for all $k \geq 3$ and the Lichnerowicz-Matsushima criterion does not apply in these cases.

It may be asked whether a Fano manifold with reductive group of automorphisms or, more generally, a Fano manifold for which Futaki criterion is fulfilled — cf. Theorem 4.12.1 below — admits Kähler-Einstein metrics. This long standing “folkloric conjecture” was supported by the following:

**Theorem 6.6.1** (G. Tian [181]). All (smooth) Fano complex surfaces whose automorphism group is reductive, i.e. all Fano complex surfaces apart from the blow-up of $\mathbb{P}^2$ at one or at two points, admit a Kähler-Einstein metric.

For the proof of this result, we refer to Tian’s original paper [181], cf. also [191], [157] (it must be observed that the assumption smooth is here essential and that the conclusion of Theorem 6.6.1 does not hold for Fano complex surfaces with orbifold singularities, [69]).

The above “folkloric conjecture” proved however to be false when G. Tian constructed the first known example of a Fano manifold of complex dimension 3, which has no non-trivial holomorphic vector field and which admits no Kähler-Einstein metric, cf. [186]: in this paper, G. Tian establishes the non-existence of a Kähler-Einstein metric by showing that the manifold contradicts the so-called weak $K$-stability condition introduced in the same paper, and formulates the conjecture that a Fano complex manifold admits Kähler-Einstein metric if and only if it is weakly $K$-stable if it has no non-trivial holomorphic vector field, [186, Conjecture 5], cf. also [186, Acknowledgement] for comments about this conjecture.

**Remark 6.6.1.** By Proposition 6.6.1 the blow-up of $\mathbb{P}^2$ at one point — which is also the first Hirzebruch surface $\mathbb{F}_1$, cf. Section 6.4 — admits no Kähler metric with constant scalar curvature, but it does admit extremal Kähler metrics as shown by E. Calabi in [45]. Calabi construction on $\mathbb{F}_1$ and the other Hirzebruch surfaces, as well as its natural extension to Hirzebruch-like ruled surfaces of any genus is the main subject of Chapter 10.

On the other hand, the issue of the existence/non-existence of extremal Kähler metrics on the blow-up of $\mathbb{P}^2$ at two points has long remained an open question until the recent works of C. Arezzo-F. Pacard [13], [14] and of C. Arezzo-F. Pacard-M. Singer [15].
Kähler-Einstein metrics on Fano manifolds

Recall that a Fano manifold is a (connected) compact complex manifold \((M, J)\) whose (real) first Chern class \(c_1(M, J)\) in \(H^2(M, \mathbb{R})\) is a Kähler class, cf. Section 1.20.

The existence issue for Kähler-Einstein metrics on Fano manifolds was completely solved by G. Tian in the case when \(m = 2\), cf. Theorem 6.6.1, but has remained a most challenging open question when \(m > 2\).

7.1. The Ricci potential of a Fano manifold

Throughout this Chapter — unless otherwise specified — \((M, J)\) will denote a Fano manifold, of complex dimension \(m \geq 2\), equipped with the Kähler class

\[
\Omega = \frac{2\pi}{k} c_1(M, J),
\]

for some \(k > 0\). We also fix a base-point \((g_0, \omega_0)\) in the space \(\mathcal{M}_\Omega\) of Kähler metrics of Kähler class \(\Omega\), which will be most often identified with \(\mathcal{M}_{\omega_0}\), where \(\mathcal{M}_{\omega_0}\) denotes the Donaldson gauge functional, cf. Section 4.1. Unless otherwise specified, any element of \(\mathcal{M}_\Omega\) will then be written unambiguously as \(\omega_0 + dd^c \phi\), with \(I_{\omega_0}(\phi) = 0\).

The total volume \(\int_M v_g\) and the mean curvature \(\int_M s_g v_g\) with respect to any metric \(g\) in \(\mathcal{M}_\Omega\) are denoted by \(V\) and \(\bar{s}\) respectively.

**Lemma 7.1.1.** For any \((g_\phi, \omega_\phi = \omega_0 + dd^c \phi)\), the mean curvature is given by

\[
\bar{s} = 2km,
\]

and the Ricci form by

\[
\rho_{g_\phi} = k \omega_\phi - \frac{1}{2} dd^c p_{g_\phi},
\]

where the Ricci potential \(p_g\), normalized by (1.19.11), has the following expression

\[
p_{g_\phi} = p_{g_0} + \log \frac{v_{g_\phi}}{v_{g_0}} + 2k(\phi - I_{\omega_0}(\phi)) - \frac{1}{V} E_{\omega_0}(\omega_\phi),
\]

where \(E_{\omega_0}\) denotes the Mabuchi K-energy centered at \(\omega_0\), as defined in Section 4.13.

**Proof.** By Proposition 1.19.1, the class of \(\rho_{g_\phi}\) in \(H^2(M, \mathbb{R})\) is \(2\pi c_1(M, J)\). Since \(2\pi c_1(M, J) = k \Omega\), the harmonic part of \(\rho_{g_\phi}\) with respect to \(g_\phi\) is then \(k \omega_\phi\). In particular, \(\bar{s} = 2\Lambda_{\omega_\phi}(k \omega_\phi) = 2km\), whereas \(\rho_\phi\) is given by (7.1.3),
where the Ricci potential \( p_{g_\phi} \), as defined by (1.19.10)-(1.19.11), is given by (1.21.3), up to an additive constant, hence by

\[
(7.1.5) \quad p_{g_\phi} = p_{g_0} + 2k \phi + \log \frac{v_{g_\phi}}{v_{g_0}} - \frac{C(\phi)}{V},
\]

where \( C(\phi) \) is constant on \( M \) and smoothly depends on \( \phi \). By integrating along \( v_{g_\phi} \), this yields

\[
(7.1.6) \quad C(\phi) = \int_M p_{g_0} v_{g_\phi} + 2k \int_M \phi v_{g_\phi} + \int_M \log \frac{v_{g_\phi}}{v_{g_0}} v_{g_\phi}.
\]

We now regard \( C \) as a function defined on \( \hat{\mathcal{M}}_{\omega_0} \) and we compute its derivative at \( \phi \) — corresponding to the metric \((g_\phi, \omega_\phi)\) in \( \mathcal{M}_\Omega \) — along any \( f \) in \( T_p \hat{\mathcal{M}}_{\omega_0} = C^\infty(M, \mathbb{R}) \). By using (3.2.4), we obtain

\[
(7.1.7) \quad (dC)_\phi(f) = -\int_M p_{g_0} \Delta_{g_\phi} f v_{g_\phi} + 2k \int_M f v_{g_\phi} - 2k \int_M \phi \Delta_{g_\phi} f v_{g_\phi} - \int_M \Delta_{g_\phi} f v_{g_\phi} + \int_M \log \frac{v_{g_\phi}}{v_{g_0}} v_{g_\phi}.
\]

By using (7.1.5) and (1.15.6) — as well as (1.19.2) and (7.1.2) — we then get

\[
(7.1.8) \quad (dC)_\phi(f) = \int_M f \Delta_{\omega_\phi} df p_{g_0} v_{g_\phi} + 2k \int_M f v_{g_\phi} \]

\[
= \int_M \Lambda_{\omega_\phi} (2k \phi - 2p_{g_\phi}) f v_{g_\phi} + 2k \int_M f v_{g_\phi} \]

\[
= -\int_M (s_{g_\phi} - \bar{s}) f v_{g_\phi} + 2k \int_M f v_{g_\phi} \]

\[
= d(E_{\omega_0} + 2k V \Pi_{\omega_0})_\phi(f).
\]

Since \( C(0) = 0 \), we deduce \( C(\phi) = E_{\omega_0}(\phi) + 2kV_{\Omega} \Pi_{\omega_0}(\phi) \).

**Remark 7.1.1.** By integrating both sides of (7.1.4) along \( v_{g_\phi} \), we get the following expression of the Mabuchi \( K \)-energy on a Fano manifold

\[
(7.1.9) \quad E_{\omega_0}(\omega_\phi) = \int_M \log \frac{v_{g_\phi}}{v_{g_0}} v_{g_\phi} + 2k \int_M \phi v_{g_\phi} + \int_M p_{g_0} v_{g_\phi},
\]

for any \( \omega_\phi = \omega_0 + dfd^c\phi \), with \( \Pi_{\omega_0}(\phi) = 0 \). By Proposition 4.2.2, \( \int_M \phi v_{g_\phi} \leq 0 \); we thus get the following majoration of \( E_{\omega_0} \):

\[
(7.1.10) \quad \frac{1}{V} E_{\omega_0}(\omega_\phi) \leq \frac{1}{V} \int_M \log \frac{v_{g_\phi}}{v_{g_0}} v_{g_\phi} + \max_M p_{g_0}.
\]

**Lemma 7.1.2.** The first variation of the Ricci potential \( p_g \) in \( \mathcal{M}_\Omega \) is given by

\[
(7.1.11) \quad \dot{p}_g = -\Delta_g f + 2k f + \frac{1}{V} \int_M p_g \Delta_g f v_g,
\]

for any \( g \) in \( \mathcal{M}_\Omega = \Pi_{\omega_0}^{-1}(0) \) and any \( f \) in \( T_g \mathcal{M}_\Omega \) (so that \( \int_M f v_g = 0 \)).
7.2. Holomorphic vector fields and Poincaré inequalities

**Definition 8.** A Kähler metric in $\mathcal{M}_\Omega$ is a *Kähler-Ricci soliton* if the Ricci potential $p_g$ is a Killing potential, i.e. if $X_f := \text{grad}_g p_g = J \text{grad}_g p_g$ is a Killing vector field with respect to $g$. Equivalently, $g$ is a Kähler-Ricci soliton if

\[ \rho = k \omega + \frac{1}{2} \mathcal{L}_X \omega, \]

for some Hamiltonian Killing (real) vector field $X$.

**Lemma 7.2.1.** For any complex 1-form $\alpha$ such that $d\alpha$ is $J$-invariant and for any metric $g$ in $\mathcal{M}_\Omega$, we have the following identity

\[ e^{p_g} \delta (e^{-p_g} D^{-} \alpha) = \frac{1}{2} \Delta_g \alpha - k \alpha + \frac{1}{2} d \langle \alpha, dp_g \rangle - \frac{1}{2} d^c \langle J \alpha, dp_g \rangle. \]

**Proof.** In (7.2.1) and throughout this paragraph, $\langle \cdot, \cdot \rangle$ denotes the *hermitian* inner product determined by $g$, which is $\mathbb{C}$-linear in the first argument and $\mathbb{C}$-anti-linear in the second one. From (1.23.10), with $\rho = k \omega - \frac{1}{2} d\bar{d} p_g$, we infer

\[
\delta D^{-} \alpha = \frac{1}{2} \Delta_g \alpha - k \alpha - \frac{1}{2} J \alpha^* \bar{d} \bar{d} p_g
= \frac{1}{2} \Delta_g \alpha - k \alpha - \frac{1}{2} \mathcal{L}_{J \alpha^*} \bar{d} p_g + \frac{1}{2} d (J \alpha^* \bar{d} p_g)
= \frac{1}{2} \Delta_g \alpha - k \alpha + \frac{1}{2} d (\alpha, dp_g) - \frac{1}{2} d^c (J \alpha, dp_g) + \frac{1}{2} dp_g \circ \mathcal{L}_{J \alpha^*} J
\]

for any 1-form $\alpha$, where $\alpha^*$ denotes the $\mathbb{C}$-linear dual of $\alpha$, defined by $g(\alpha^*, Z) = \alpha(Z)$ for any (complex) vector field $Z$, and $g$ is $\mathbb{C}$-bilinearly extended to complex vector fields. The rhs of (7.2.1) is then equal to $\delta D^{-} \alpha - \frac{1}{2} dp_g \circ \mathcal{L}_{J \alpha^*} J$, whereas, by (1.23.7), $- \frac{1}{2} dp_g \circ \mathcal{L}_{J \alpha^*} J = D^{-} \alpha (\cdot, \text{grad}_g p_g)$. Now, $d\alpha$ is $J$-invariant if and only if $D^{-} \alpha$ is symmetric (easy verification). We thus get: $\delta D^{-} \alpha - \frac{1}{2} dp_g \circ \mathcal{L}_{J \alpha^*} J = \delta D^{-} \alpha + D^{-} \alpha (\text{grad}_g p_g, \cdot) = e^{p_g} \delta (e^{-p_g} D^{-} \alpha)$. □

**Lemma 7.2.2.** For any Kähler metric $g$ in $\mathcal{M}_\Omega$, of Ricci potential $p_g$, and for any complex-valued function $F$, we have

\[ e^{p_g} \delta (e^{-p_g} D^{-} \partial F) = \partial (\hat{\mathbb{D}}_g^+ F - k F), \]

where $\hat{\mathbb{D}}_g^+$ denote the Futaki twisted Laplacian defined by:

\[ \hat{\mathbb{D}}_g^+ = e^{p_g} \partial^* (e^{-p_g} \partial F), \]

cf. [86, page 40].

---

**Proof.** Direct consequence of (7.1.4), of (4.10.7) and of (1.19.12). □
7. KÄHLER-EINSTEIN METRICS ON FANO MANIFOLDS

Proof. Direct application of (7.2.1), with \( \alpha = \bar{\partial}F \), via the alternative formulation of \( D^+_g \):

\[
D^+_g F = \Box_g F + \langle \bar{\partial} F, dp_g \rangle = \frac{1}{2} \Delta_g F + \langle \bar{\partial} F, dp_g \rangle.
\]

\[\square\]

Proposition 7.2.1 ([86] Theorem 2.4.3, [190] Lemma 3.1). Let \( g \) be any Kähler metric in \( \mathcal{M}_1 \), and denote by \( p_g \) its Ricci potential. Then, for any complex function \( F \) such that \( \int_M F e^{-p_g} v_g = 0 \), we have:

\[
\int_M |\bar{\partial}F|^2 e^{-p_g} v_g - k \int_M |F|^2 e^{-p_g} v_g \geq 0,
\]

with equality if and only if \( F \) is a holomorphic potential with respect to \( g \), cf. Definition 2. In particular, for any real function \( f \) such that \( \int_M f e^{-p_g} v_g = 0 \), we have:

\[
\int_M |df|^2 e^{-p_g} v_g - 2k \int_M f^2 e^{-p_g} v_g \geq 0,
\]

with equality if and only if \( f \) is a Killing potential with respect to \( g \).

Proof. The Futaki twisted laplacian \( D^+_g \) is a linear elliptic second order differential operator, defined on the space \( C^\infty(M, \mathbb{C}) \) of complex-valued functions. From (7.2.3), we readily infer that it is self-adjoint with respect to the volume form \( e^{-p_g} v_g \), i.e. \( D^+_g \) satisfies:

\[
\int_M \langle D^+_g F_1, F_2 \rangle e^{-p_g} v_g = \int_M \langle F_1, D^+_g F_2 \rangle e^{-p_g} v_g,
\]

for any complex functions \( F_1, F_2 \), and that

\[
\int_M \langle \bar{\partial} F, F \rangle e^{-p_g} v_g = \int_M |\bar{\partial}F|^2 e^{-p_g} v_g,
\]

which implies that \( D^+_g \) is semi-positive and that its kernel is the space of holomorphic functions, hence the space \( \mathbb{C} \) of constant functions, since \( M \) is compact. We then have

\[
C^\infty(M, \mathbb{C}) \subset L^2(M, \mathbb{C}; e^{-p_g} v_g) = \bigoplus_{r=0}^{\infty} E_{\lambda_r},
\]

where \( 0 = \lambda_0 < \lambda_1 < \ldots < \lambda_r < \ldots \) denotes the eigenvalues of \( D^+_g \), whereas the sum is a Hilbert sum with respect to the hermitian product

\[
\langle F_1, F_2 \rangle_{p_g} := \int_M \langle F_1, F_2 \rangle e^{-p_g} v_g,
\]

with \( E_0 = \mathbb{C} \). In particular

\[
D_0 := C^\infty(M, \mathbb{C}) \cap \bigoplus_{r=1}^{\infty} E_{\lambda_r}
\]

for \( D_0 \) and \( D \).

\[
D_0 := \{ F \in C^\infty(M, \mathbb{C}) \mid \int_M F e^{-p_g} v_g = 0 \}.
\]

From (7.2.2), we infer that

\[
\int_M |D^- \bar{\partial}F|^2 e^{-p_g} v_g = \lambda_r(\lambda_r - k) \int_M |F|^2 e^{-p_g} v_g,
\]

which implies that

\[
D^- \bar{\partial}F = 0
\]

and, therefore, \( \lambda_r(\lambda_r - k) \int_M |F|^2 e^{-p_g} v_g = 0 \). This completes the proof.

\[\square\]
for any eigenvalue $\lambda_r$ and any (complex) eigenfunction $F$ of $\mathbb{D}_g^+$ relative to $\lambda_r$. This implies that any positive eigenvalue $\lambda_r$ of $\mathbb{D}_g^+$ satisfies $\lambda_r \geq k$ and that $k$ is actually an eigenvalue of $\mathbb{D}_g^+$ if and only if the space of holomorphic potentials $F$ with respect to $g$, normalized by $\int_M F e^{-p_g} v_g = 0$, is not reduced to $\{0\}$; this is then identified with the $k$-eigenspace of $\mathbb{D}_g^+$, cf. Section 2.5. It follows that the restriction of $\mathbb{D}_g^+ - k \text{Id}$ to $\mathcal{D}_0$ is semi-positive and that its kernel is either $\{0\}$ or $E_{\lambda_1}$, i.e. the space of holomorphic potentials $F$ normalized by $\int_M F e^{-p_g} v_g$, if the latter is not reduced to $\{0\}$, i.e. if $\lambda_1 = k$. From (7.2.8), we then infer

\begin{equation}
(7.2.13)
\int_M \left( \left( \mathbb{D}_g^+ F - k F \right), F \right) e^{-p_g} v_g = \int_M \left| \partial F \right|^2 e^{-p_g} v_g - k \int_M |F|^2 e^{-p_g} v_g \geq 0,
\end{equation}

for any complex function $F$ such that $\int_M F e^{-p_g} v_g = 0$, with equality if and only if $F$ is holomorphic potential. We thus get the first part of Proposition 7.2.1. If $f$ is real, we get (7.2.6) by observing that $\left| \partial f \right|^2 = \frac{1}{2} |df|^2$, and equality holds if and only if $X = \text{grad}_g f$ is (real) holomorphic, if and only if $f$ is a Killing potential, cf. Section 2.6. \qed

**Remark 7.2.1.** In the above argument, the complex function $F$ can be conveniently regarded as a section of the trivial complex line bundle $L = M \times \mathbb{C}$ equipped with its natural holomorphic structure and with the hermitian inner product defined by (7.2.10), for which $|1|^2_{p_g} = \int_M e^{-p_g} v_g$. In particular, the curvature form of the corresponding Chern connection $\nabla$ is $k \omega - \rho$. With respect to this hermitian inner product, the (hermitian) adjoint of $\bar{\partial}$ is $\bar{\partial}^*_p := e^{p_g} \bar{\partial}^* e^{-p_g}$ and $\mathbb{D}_g^+$ is then the usual Dolbeault laplacian $\square^L F = (\bar{\partial}^*)^L \bar{\partial} F$, as defined in Section 1.16.

**Proposition 7.2.2.** For any Kähler metric $g$ in $\mathcal{M}_\Omega$, of Ricci potential $p_g$, denote by $\mathbb{D}_g$ the operator on $C^\infty(M, \mathbb{R})$ defined by

\begin{equation}
(7.2.14)
\mathbb{D}_g f = e^{p_g} \delta(e^{-p_g} f) = \Delta_g f + (df, dp_g),
\end{equation}

for any real function $f$. Denote by $\mathbb{D}_0^R = C^\infty(M, \mathbb{R}) \cap \mathcal{D}_0$ the space of smooth real functions $f$ such that $\int_M f e^{-p_g} v_g = 0$. Then, $\mathbb{D}_g - 2k \text{Id}$ preserves $\mathbb{D}_0^R$, its restriction to $\mathbb{D}_0^R$ is semi-positive and, for any $f$ in $\mathbb{D}_0^R$,

\begin{equation}
(7.2.15)
\mathbb{D}_g f - 2k f = 0,
\end{equation}

if and only if $f$ is a Killing potential with respect to $g$.

**Proof.** The operator $\mathbb{D}_g$ defined by (7.2.14) is clearly self-adjoint with respect to the volume form $e^{-p_g} v_g$. Moreover, since $\int_M (\mathbb{D}_g f - 2k f) f e^{-p_g} v_g = \int_M |df|^2 e^{-p_g} v_g - 2k \int_M f^2 e^{-p_g} v_g$, it follows from the second part of Proposition 7.2.1) that the restriction of $\mathbb{D}_g - 2k \text{Id}$ to $\mathbb{D}_0^R$ and that its kernel is the space of Killing potentials $f$ with respect to $g$ normalized by $\int_M f e^{-p_g} v_g$. \qed

**Remark 7.2.2.** By the proof of Proposition 7.2.1, we know that a complex function $F$ such that $\int_M F e^{-p_g} v_g = 0$ is a holomorphic potential with respect to $g$ if and only if $\mathbb{D}_g^+ F - k F = 0$, i.e.

\begin{equation}
(7.2.16)
\Delta_g F - 2k F + (dF, dp_g) - i(df, dp_g) = 0.
\end{equation}
A real function \( f \) is then a Killing potential, i.e. a real holomorphic potential, with respect to \( g \) if and only if (7.2.15) is satisfied as well as \((d\bar{c} f, dp_g) = 0\). Notice that the latter also reads \( \mathcal{L}_{\text{grad}}f p_g = 0 \) and is then automatically satisfied if \( f \) is a Killing potential since \( \text{grad} f \) is then a Killing vector field. Conversely, any real function \( f \) satisfying \( \int_M f e^{-p_g} v_g = 0 \) and (7.2.15) satisfies \( \int_M |df|^2 e^{-p_g} v_g = 0 \) and (7.2.16) shows that, up to the.

**Proposition 7.2.3.** Let \( X \) be any (real) holomorphic vector field on \( M \). For any Kähler metric \( g \) in \( \mathcal{M}_\Omega \), of Ricci potential \( p_g \), denote by \( f^X_g \) the real potential of \( X \) with respect to \( g \), normalized by \( \int_M f^X_g v_g = 0 \). Denote by \( \mathcal{F}_\Omega : h \to \mathbb{R} \) the Futaki character determined by \( \Omega \). Then

\[
(7.2.17) \quad \frac{1}{V} \mathcal{F}_\Omega(X) = -2k \int_M f^X_g e^{-p_g} v_g.
\]

**Proof.** Denote by \( F = f^X_g + i h^X_g \) the complex potential of \( X \), normalized by \( \int_M F v_g = 0 \). From (7.2.16) we infer

\[
(7.2.18) \quad \Delta_g F - 2k F + (dF, dp_g) - i(d\bar{c} F, dp_g) = C,
\]

for some (complex) constant \( C \). By integrating over \( M \) along \( v_g \), and by taking into account that \( \delta^c d = 0 \), we get

\[
(7.2.19) \quad C V = \int_M (dF, dp_g) v_g = \int_M F \Delta_g v_g = \int_M F (s_g - \bar{s}) v_g = \mathcal{F}_\Omega(X) - i\mathcal{F}_\Omega(JX).
\]

By integrating along \( e^{-p_g} v_g \), we get instead:

\[
(7.2.20) \quad C \int_M e^{p_g} v_g = -2k \int_M F e^{-p_g} v_g.
\]

By comparing these two expressions of \( C \), we get (7.2.17).

**Remark 7.2.3.** If, instead, the real potential of \( X \), call it then \( \tilde{f}^X_g \), is normalized by \( \int_M \tilde{f}^X_g e^{-p_g} v_g \), a similar argument shows that

\[
(7.2.21) \quad \mathcal{F}_\Omega(X) = 2k \int_M \tilde{f}^X_g v_g.
\]

**Remark 7.2.4.** In view of (7.5.5) below, (7.2.17) shows that, up to the factor \(-2k V \), \( \mathcal{F}_\Omega(X) \) is equal to the derivative of the Ding functional, introduced in Section 7.5 below, along the lift \( \hat{X} \) of \( X \) on \( \mathcal{M}_\Omega \), i.e. that \( \mathcal{F}_\Omega = 0 \) if and only if the Ding functional is constant on each orbit of \( H(M, J) \), cf. Remark 4.12.4 in Section 4.12.

**Proposition 7.2.4.** Let \( g \) be a Kähler metric in \( \mathcal{M}_\Omega \) and suppose that \( g \) is a Kähler-Ricci soliton, as in Definition 8. As usual, denote by \( K(M, g) \) the identity component of the group of isometries of \((M, g)\) and by \( H(M, J) \) the identity component of the group of automorphisms of \((M, J)\). Then, \( K(M, g) \) is a maximal compact subgroup of \( H(M, J) \).
7.3. The modified Monge-Ampère equation

Proof. The statement is similar to Theorem 3.5.1 and the proof is also quite similar, even simpler since \( h = h_{\text{red}} \), as \( M \) is Fano, hence simply-connected. Consider the conjugate Futaki twisted laplacian \( \mathbb{D}_g^+ \), defined by 
\[
\mathbb{D}_g^+ F = \mathbb{D}_g^+ F = e^{ps} \partial^* (e^{-ps} \partial F).
\]
Like \( \mathbb{D}_g^+ \), \( \mathbb{D}_g^- \) is a second order elliptic operator on \( C^\infty (M, \mathbb{C}) \), self-adjoint and semi-positive with respect to the volume form \( e^{-ps} v_g \), with the same spectrum \( 0 = \lambda_0 < \lambda_1 < \ldots < \lambda_r < \ldots \) as \( \mathbb{D}_g^+ \). On the space \( \mathcal{D}_g \) of complex functions \( F \) satisfying \( \int_M e^{-ps} v_g = 0 \), cf. (7.2.11), the two operators \( \mathbb{D}_g^+ - k \text{Id} \) and \( \mathbb{D}_g^- - k \text{Id} \) are then semi-positive, of the form
\[
\begin{align*}
\mathbb{D}_g^+ F - k F &= \frac{1}{2} e^{ps} \delta (e^{-ps} dF) - k F + \frac{i}{2} \mathcal{L}_{\text{grad}_g p_g} F, \\
\mathbb{D}_g^- F - k F &= \frac{1}{2} e^{ps} \delta (e^{-ps} dF) - k F - \frac{i}{2} \mathcal{L}_{\text{grad}_g p_g} F,
\end{align*}
\]
with \( \delta \) the operator on \( C^\infty (M, \mathbb{C}) \), defined by 
\[
\delta = \frac{1}{2} \partial^* \partial = \frac{1}{2} \Delta_g.
\]
Moreover, since \( J \text{grad}_g p_g \) is a Killing vector field, \( \mathbb{D}_g^+ - k \text{Id} \) and \( \mathbb{D}_g^- - k \text{Id} \) commute, cf. Proposition 3.3.2. It follows that \( \mathbb{D}_g^- - k \text{Id} \) acts on the kernel \( \ker (\mathbb{D}_g^+ - k \text{Id}) \) of \( \mathbb{D}_g^+ - k \text{Id} \) as a semi-positive hermitian operator, and coincides there with \( \mathbb{D}_g^+ - k \text{Id} \) on \( \ker (\mathbb{D}_g^+ - k \text{Id}) \). In particular, the kernel of this action in \( \ker (\mathbb{D}_g^+ - k \text{Id}) \cong h \) is \( \mathfrak{t} \oplus \mathfrak{j}t \), and \( h \) then splits as
\[
(7.2.23) \quad h = \mathfrak{t} \oplus \mathfrak{j}t \oplus \oplus_{\lambda > 0} h^{(\lambda)},
\]
with \( h^{(\lambda)} = \{ X \in h | [ J \text{grad}_g p_g, X] \lambda JX \} \). Notice that this decomposition is formally identical to (3.4.5), with \( J \text{grad}_g p_g \) instead of \( K = J \text{grad}_g s_g \). The rest of the argument is the same as for Theorem 3.5.1. \( \square \)

7.3. The modified Monge-Ampère equation

In view of Lemma 7.1.1, the Einstein condition \( \rho_{g_0} = k \omega_0 \), which is equivalent to the condition \( p_{g_0} \equiv 0 \), can then be rewritten as the following Monge-Ampère equation:

\[ m \quad (\omega_0 + dd^c \phi)^m = e^{-p_{g_0} - 2k \phi + \frac{1}{V} E_{\omega_0} (\omega_0)} \omega_0^m, \]

where, we recall, \( \phi \) is normalized by \( \mathbb{L}_{\omega_0} (\phi) = 0 \). In particular, for any solution to (7.3.1) in \( \mathbb{L}_{\omega_0}^1 (0) \), the Mabuchi K-energy of \( \omega_0 \) is given by

\[
(7.3.2) \quad \frac{1}{V} E_{\omega_0} (\omega_0) = - \log \frac{1}{V} \int_M e^{-p_{g_0} - 2k \phi} v_{g_0} = \log \frac{1}{V} \int_M e^{p_{g_0} + 2k \phi} v_{g_0}.
\]

The continuity method introduced by T. Aubin — cf. Section 1.20 — consists in replacing the Einstein equation \( \rho_{g_0} = k \omega_0 \) by the following family of equations:

\[ m \quad \rho_{g_0} = k (\omega_0 + td^c \phi) = k (\omega_0 - (1 - t) d^c \phi) = tk \omega_0 + (1 - t) k \omega_0, \]
parametrized by a real parameter \( t \) in the closed interval \([0, 1]\); equivalently, for each \( t \) in \([0, 1]\), the Ricci potential has the following expression:

\[
p_{g_0} = 2k(1-t) \phi - \frac{2k(1-t)}{V} \int_M \phi v_{g_0},
\]

for \( \omega_\phi = \omega_0 + dd^c \phi \) in \( \mathcal{M}_\Omega \). In view of Lemma 7.1.1, the latter equation is equivalent to the following modified Monge–Ampère equation:

\[
(\omega_0 + dd^c \phi)^m = e^{-p_{g_0} - 2tk \phi} + c(t, \phi) \omega_0^m,
\]

where \( c(t, \phi) \) is defined by

\[
c(t, \phi) = \frac{1}{V} E_{\omega_0}(\omega_\phi) - \frac{2k(1-t)}{V} \int_M \phi v_{g_0},
\]

and \( \phi \) is normalized by \( \mathbb{I}_{\omega_0}(\phi) = 0 \). From (7.3.5), we also infer

\[
c(t, \phi) = -\log \frac{1}{V} \int_M e^{-p_{g_0} - 2tk \phi} v_{g_0} = \log \frac{1}{V} \int_M e^{p_{g_0} + 2tk \phi} v_{g_0}.
\]

For \( t = 1 \), (7.3.3) is the initial Kähler-Einstein equation. For \( t = 0 \), (7.3.3) reduces to the equation

\[
\rho_{g_0} = k \omega_0,
\]

and (7.3.5) specializes to

\[
(\omega_0 + dd^c \phi)^m = e^{-p_{g_0} + c(0, \phi)} \omega_0^m,
\]

with \( \mathbb{I}_{\omega_0}(\phi) = 0 \), where, by (7.3.6)-(7.3.7), \( c(0, \phi) \) is given by

\[
c(0, \phi) = \frac{1}{V} E_{\omega_0}(\omega_\phi) - \frac{2k}{V} \int_M \phi v_{g_0} = -\log \frac{1}{V} \int_M e^{-p_{g_0}} v_{g_0}.
\]

**Proposition 7.3.1.** For any chosen element \( \omega_0 \) in \( \mathcal{M}_\Omega \), the equation (7.3.8) has a unique solution, \( \phi_0 \), in \( \mathbb{I}_{\omega_0}^{-1}(0) \) and we then have

\[
E_{\omega_0}(\omega_\phi_0) \leq 0,
\]

for \( \omega_{\phi_0} = \omega_0 + dd^c \phi_0 \).

**Proof.** The existence and uniqueness of a solution \( \phi_0 \) of (7.3.8), equivalently of the Monge-Ampère equation (7.3.9), is a direct consequence of the Calabi-Yau theorem 1.20.1. From (7.3.10), we infer

\[
\frac{1}{V} E_{\omega_0}(\omega_\phi_0) = \frac{2k}{V} \int_M \phi_0 v_{g_0} - \log \frac{1}{V} \int_M e^{-p_{g_0}} v_{g_0},
\]

where the rhs is the sum of two terms which are both non-positive: the first one by Proposition 4.2.2, as \( \mathbb{I}_{\omega_0}(\phi_0) = 0 \), the second one by the *Jensen inequality*, since \( \int_M p_{g_0} v_{g_0} = 0 \).

\[\text{fnjensen}\]

1 In general, any real function \( f \) defined on a compact oriented riemannian \((M, g)\), of volume form \( v_g \) and of total volume \( V = \int_M v_g \), satisfies the *Jensen inequality*:

\[
\frac{1}{V} \int_M f v_g \leq \log \frac{1}{V} \int_M e^f v_g,
\]

with equality if and only if \( f \) is constant. This is easily deduced from the concavity of the graph of the log function, cf. also the footnote 3 of the page 50.
Remark 7.3.1. Denote by \( \mathcal{M}_\Omega^+ \) the subspace of elements of \( \mathcal{M}_\Omega \) whose Ricci form is positive definite, hence of the form \( k \omega \), where \( \omega \) is the Kähler form of an element of \( \mathcal{M}_\Omega \). Then, \( \mathcal{M}_\Omega^+ \) coincides with the space of solutions \( (g_{\omega_0}, \omega_0 \omega = \omega_0 + dd^c \phi_0) \) of (7.3.8)-(7.3.9) when \( \omega_0 \) runs over all elements of \( \mathcal{M}_\Omega \). By changing the notation, Proposition 7.3.1 has then the following consequence: for any \( \omega \) in \( \mathcal{M}_\Omega \), there exists \( \omega^+ \) in \( \mathcal{M}_\Omega^+ \) such that

\[
E_{\omega^+}(\omega^+) \leq E_{\omega}(\omega),
\]

for any base-point \( \omega_0 \) in \( \mathcal{M}_\Omega \). Indeed, by (7.3.11), we have: \( E_{\omega_0}(\omega^+) - E_{\omega_0}(\omega) = E_{\omega}(\omega^+) - E_{\omega}(\omega) \leq 0 \), if \( \omega \) is chosen as the base-point in (7.3.8)-(7.3.9) and \( \omega^+ = \omega_{\omega_0} \) is then chosen to be the unique solution to (7.3.8)-(7.3.9) of base-point \( \omega \). In particular, if the \( K \)-energy \( E_{\omega_0} \) happens to be bounded from below on \( \mathcal{M}_\Omega^+ \), it is bounded from below on the whole space \( \mathcal{M}_\Omega \).

Proposition 7.3.2 ([20], [23] Proposition 4.4.1). There exists \( T \) in \( (0, 1] \) and a unique smooth curve \( \{ \phi_t \}_{t \in [0, T]} \) in \( \mathbb{L}^{-1}_w(0) \) such that, for each \( t \) in \( [0, T] \), \( \phi_t \) is a solution to (7.3.5) and \( \{ \phi_t \} \) extends to \([0, T') \) for no \( T' > T \) in \((0, 1]\).

Proof. (Sketch) Denote by \( \mathcal{H}_0 \) the subspace of \( C^\infty(M, \mathbb{R}) \) defined by \( \mathcal{H}_0 = \{ f \mid \frac{1}{2} \int_M c^f v_{g_0} = 1 \} \) and let \( \mathcal{V} \) the map from \([0, 1] \times \mathbb{L}^{-1}_w(0) \) to \( \mathcal{H}_0 \) defined by

\[
\mathcal{V}(t, \phi) = \log \frac{v_{g_0}}{v_{g_0}} + p_{g_0} + 2t k \phi - \log \frac{1}{V} \int_M e^{p_{g_0} + 2tk \phi} v_{g_0}.
\]

Then, \( \phi \) in \( \mathbb{L}^{-1}_w(0) \) is a solution to (7.3.5) if and only if \( \mathcal{V}(t, \phi) = 0 \). Moreover, by (3.2.4), for any fixed \( t \) the derivative along the second variable of \( \mathcal{V} \) at \( (t, \phi) \) is the linear map

\[
f \mapsto -(\Delta_{g_0} f - 2tk f) + \frac{\int_M (\Delta_{g_0} f - 2tk f) e^{p_{g_0} + 2tk \phi}}{\int_M e^{p_{g_0} + 2tk \phi}} v_{g_0},
\]

from \( T_{\phi} \mathbb{L}^{-1}_w(0) = \{ f \mid \int_M f v_{g_0} = 0 \} \) to \( T_{\mathcal{V}(t, \phi)} \mathcal{H}_0 = \{ \psi \mid \int_M \psi e^{\mathcal{V}(t, \phi)} v_{g_0} = 0 \} \). If \( (t, \phi) \) is a solution to (7.3.5), so that \( \mathcal{V}(t, \phi) = 0 \), this then reduces to

\[
f \mapsto -(\Delta_{g_0} f - 2tk f) + \frac{1}{V} \int_M (\Delta_{g_0} f - 2tk f) v_{g_0},
\]

from \( T_{\phi} \mathbb{L}^{-1}_w(0) = \{ f \mid \int_M f v_{g_0} = 0 \} \) to \( T_{\mathcal{V}(t, \phi)} \mathcal{H}_0 = \{ \psi \mid \int_M \psi v_{g_0} = 0 \} \). For each \( t \) in \([0, 1]\), this map is an isomorphism of Fréchet spaces. This is because, by (7.3.3),

\[
(7.3.18) \quad r_{g_0} > tk g_0,
\]

for any solution \( (t, \phi) \) of (7.3.5), with \( t \) in \([0, 1]\), so that, by Proposition 3.6.1, \( 2tk \) is smaller than the smallest positive eigenvalue of \( \Delta_{g_0} \). This implies that the map \( f \mapsto \Delta_{g_0} f - 2tk f \) is then a Fréchet isomorphism from \( \{ f \mid \int_M f v_{g_0} = 0 \} \) to itself, hence that the map (7.3.17) is a Fréchet isomorphism from \( \{ f \mid \int_M f v_{g_0} = 0 \} \) to \( \{ \psi \mid \int_M \psi v_{g_0} = 0 \} \). By introducing appropriate Hölder spaces and by using the ellipticity of \( \Delta_{g_0} \), cf. Section 1.20 and Chapter 5, we infer: For any solution \( (t, \phi) \) of (7.3.5), \( t \in [0, 1) \), the map
\( \hat{V} : (t, \phi) \mapsto (t, V) \) induces an isomorphism from a neighborhood of \((t, \phi)\) in \((0, 1) \times \mathbb{I}_{\omega_0}^{-1}(0)\) to some neighborhood of \((t, 0)\) in \((0, 1) \times \mathcal{H}_0\).

In particular, by specializing to \(t = 0\), we get \(\varepsilon \in (0, 1]\) and a unique smooth curve \(\phi_t, t \in [0, \varepsilon]\), in \(\mathbb{I}_{\omega_0}^{-1}(0)\), whose image by \(\hat{V}\) is the curve \((t, 0)\) in \([0, 1] \times \mathcal{H}_0\). If the curve \(\phi_t\) cannot be extended to \(t = \varepsilon\), we get Proposition 7.3.2, with \(T = \varepsilon\). If the curve \(\phi_t\) smoothly extends to \(\varepsilon\), then \(\phi_{\varepsilon}\) is a solution to (7.3.5). By the above, we again get an isomorphism of a neighborhood of \((\varepsilon, \phi_{\varepsilon})\) in \([0, 1] \times \mathbb{I}_{\omega_0}\) with a neighborhood of \((\varepsilon, 0)\) in \([0, 1] \times \mathcal{H}_0\), hence a unique smooth extension of the previous curve to \([0, \varepsilon')\), for some \(\varepsilon' > \varepsilon\) in \([0, 1]\). This completes the proof of Proposition 7.3.2, by choosing \(T\) equal to the upper bound of the \(\varepsilon\)'s.

**Definition 9.** The smooth curve \(\{\phi_t\}_{t \in [0, T]}\) in \(\mathbb{I}_{\omega_0}^{-1}(0)\) determined by Proposition 7.3.2 is called the maximal solution of the modified Monge-Ampère equation (7.3.5).

**Remark 7.3.2.** It follows from the proof of Proposition 7.3.2 that the maximal solution \(\{\phi_t\}_{t \in [0, T]}\) does not extend to \(T\) unless, possibly, if \(T = 1\). If \(T = 1\) and if \(\{\phi_t\}_{t \in [0, T]}\) does extend to \(T = 1\), then \(\phi_T = \phi_1\) is the relative Kähler potential with respect of \(\omega_0\) of a Kähler-Einstein metric. It was shown by S.-T. Yau in [198], cf. also Lemma 3.2 and Lemma 3.3 in [185], that this happens if and only if \(|\phi_t|_{C^0} = \max (\sup_M \phi_t, - \inf_M \phi_t)\) is uniformly bounded (Notice that, by Proposition 4.2.2, the condition \(\mathbb{I}_{\omega_0}(\phi_t) = 0\) implies \(\sup_M \phi_t \geq 0\) and \(\inf_M \phi_t \leq 0\)).

The maximal solution \(\{\phi_t\}_{t \in [0, T]}\) of (7.3.5) is determined by

\[
(\omega_0 + dd^c \phi_t)^m = e^{-p_{g_0} - 2tk \phi_t + c(t, \phi_t)} \omega_0^m,
\]

where \(\phi_t\) is normalized by the Donaldson condition

\[
\mathbb{I}_{\omega_0}(\phi_t) = 0,
\]

for each \(t \in [0, T]\), and where, by Lemma 7.1.1, the constant \(c(t, \phi_t)\) has therefore the following expression:

\[
c(t, \phi_t) = \frac{1}{V} E_{\omega_0}(\omega_{\phi_t}) - \frac{2k(1 - t)}{V} \int_M \phi_t v_{g_{\phi_t}}.
\]

Notice that (7.3.19) also implies the following tautological expression of \(c(t, \phi_t)\):

\[
c(t, \phi_t) = - \log \frac{1}{V} \int_M e^{-p_{g_0} - 2tk \phi_t} v_{g_0} = \log \frac{1}{V} \int_M e^{p_{g_0} + 2tk \phi_t} v_{g_{\phi_t}}.
\]

**Proposition 7.3.3.** Along the maximal solution \(\{\phi_t\}_{t \in [0, T]}\) of (7.3.19), \(c(t, \phi_t)\) is a non increasing function of \(t\). In particular

\[
c(t, \phi_t) \leq c(0, \phi_0) = - \log \frac{1}{V} \int_M e^{-p_{g_0}} v_{g_0} \leq 0.
\]

**Proof.** By differentiating both sides of (7.3.19), as functions of \(t\), and by using (3.2.4), we get

\[
\Delta_{g_{\phi_t}} \phi_t = 2k \phi_t + 2tk \dot{\phi}_t - \dot{c}(t, \phi_t).
\]
Since $L_{\omega_t} (\phi_t) = 0$ for any $t$ in $[0, T)$, we have $\int_M \phi_t v_{g_{\phi_t}} = 0$, whereas, by Proposition 4.2.2, $\int_M \phi_t v_{g_{\phi_t}} \leq 0$. We thus infer from (7.3.24):

$$\dot{c}(t, \phi_t) = \frac{2k}{V} \int_M \phi_t v_{g_{\phi_t}} \leq 0.$$  

We thus get (7.3.23), where the last inequality is deduced from the Jenseins inequality and $\int_M p \phi_0 v_{g_0} = 0$.

**Proposition 7.3.4** (23 Theorem (5.7)). Along the maximal solution $\{\phi_t\}_{t \in [0, T)}$ of (7.3.5), $\frac{1}{V} E_{\omega_t}(\omega_{\phi_t})$ is a non-increasing function of $t$. In particular,

$$\frac{1}{V} E_{\omega_t}(\omega_{\phi_t}) \leq \frac{1}{V} E_{\omega_0}(\omega_{\phi_0}) \leq 0,$$

for any $t$ in $[0, T)$. If, moreover, the $K$-energy $E_{\omega_0}$ is bounded from below on $\mathcal{M}_\Omega$, then $T = 1$.

**Proof.** By (7.3.21), we have:

$$\frac{1}{V} E_{\omega_t}(\omega_{\phi_t}) = \frac{2k(1-t)}{V} \int_M \phi_t v_{g_{\phi_t}} + c(t, \phi_t),$$

along $\{\phi_t\}$. By Proposition 7.3.3, $c(t, \phi_t)$ is non-increasing and we now check that $\int_M \phi_t v_{g_{\phi_t}}$ is also non-increasing. Since $L_{\omega_0}(\phi_t) = 0$, we have $\int_M \phi_t v_{g_{\phi_t}} = 0$; by (3.2.4), the derivative of $\int_M \phi_t v_{g_{\phi_t}}$ is then equal to $-\int_M \phi_t \Delta g_{\phi_t} \phi_t v_{g_{\phi_t}}$. By using (7.3.24), this can be rewritten as

$$-\frac{1}{2k} \int_M \Delta g_{\phi_t} \phi_t (\Delta g_{\phi_t} - 2kt) \phi_t v_{g_{\phi_t}} = -\frac{1}{2k} \int_M \left( (\Delta g_{\phi_t} - 2kt) \phi_t \right)^2 v_{g_{\phi_t}}$$

$$- t \int M \phi_t (\Delta g_{\phi_t} - 2kt) \phi_t v_{g_{\phi_t}}. $$

By (7.3.18) and Proposition 3.6.1, the self-adjoint operator $\Delta g_{\phi_t} - 2kt$, acting on the space of real functions of mean value zero with respect to $v_{g_{\phi_t}}$, is positive definite for any $t$ in $[0, T)$, so that

$$- t \int M \phi_t (\Delta g_{\phi_t} - 2kt) \phi_t v_{g_{\phi_t}} \leq 0,$$

whereas we evidently have:

$$-\frac{1}{2k} \int_M (\Delta g_{\phi_t} - 2kt) \phi_t^2 \phi_t v_{g_{\phi_t}} \leq 0.$$  

This completes the proof of the first assertion, hence of the first inequality in (7.3.26) (the last one is (7.3.11)).

Assume that $E_{\omega_0} \geq C$ on $\mathcal{M}_\Omega$ for some positive real number $C$ and assume, for a contradiction, that $T < 1$. From (7.3.27) and Proposition 7.3.3, we infer

$$-\frac{1}{V} \int_M \phi_t v_{g_{\phi_t}} \leq c(0, \phi_0) - C \frac{2k(1-t)}{2k(1-t)} \leq c(0, \phi_0) - C \frac{2k(1-t)}{2k(1-T)}.$$

It then follows from Proposition 7.3.5 below that $\phi_t$ would extend to a solution $\phi_T$ of (7.3.5) for $t = T$, which contradicts the definition of $T$, cf. Remark 7.3.2 above.

The following statement is essentially Proposition (3.6) in (23):

**Proposition 7.3.5.** If either $-\frac{1}{V} \int_M \phi_t v_{g_{\phi_t}}$ or $\frac{1}{V} \int M \phi_t v_{g_0}$ is uniformly bounded along the maximal solution $\{\phi_t\}_{t \in [0, T)}$ of the Monge-Ampère equation, then $T = 1$ and $\phi_t$ extends to a solution $\phi_1$ of (7.3.5) for $t = 1$, which is then the relative Kähler potential of a Kähler-Einstein metric in $\mathcal{M}_\Omega$. 


Proof. By Remark 7.3.2, it is sufficient to show that a uniform upper bound of \(-\frac{1}{V} \int_M \phi_t \, v_{g_0} \) — equivalently, of \( \frac{1}{V} \int_M \phi_t \, v_{g_0} \) by Proposition 4.2.2, since \( I_{\omega_0}(\phi_t) = 0 \) — provides a uniform upper bound of \( \max_M \phi_t \) and of \(-\inf_M \phi_t\), hence a uniform upper bound of \( \|\phi_t\|_{C^0} = \inf(\max \phi_t, -\min \phi_t) \), the \( C^0 \)-norm of \( \phi_t \). This requires general arguments involving the Green functions of metrics in \( \mathcal{M}_\Omega \). In general, for any (connected) compact Riemannian manifold \((M, g)\) of dimension \( n > 1 \), the Green function \( G_g \) of \( g \) is the kernel of the Green operator \( G \) defined in Section 1.1.7 and is then determined by:

\[
\text{green-def-fano} \quad (7.3.29) \quad f(x) = \frac{1}{V} \int_M f \, v_g + \int_M G_g(x, \cdot) \Delta_g f \, v_g,
\]

and

\[
\text{green-norm} \quad (7.3.30) \quad \int_M G_g(x, \cdot) \, v_g = 0,
\]

for any (smooth) function \( f \) and any point \( x \) on \( M \), cf. e.g. [19, Chapter 4]. Let \((g_\phi, \omega_\phi = \omega_0 + dd^c \phi)\) be any element of \( \mathcal{M}_\Omega \). Then, by using (1.15.6), we get

\[
\text{minDeltaphit} \quad (7.3.31) \quad \Delta_{g_\phi} \phi = -\Lambda_{\omega_\phi} dd^c \phi = m - \Lambda_{\omega_\phi} \omega_0 \leq m,
\]

and

\[
\text{maxDeltaphit} \quad (7.3.32) \quad -\Delta_{g_\phi} \phi = \Lambda_{\omega_\phi} dd^c \phi = m - \Lambda_{\omega_\phi} \omega_0 \leq m,
\]

cf. Remark 1.20.1). Let \( x_m \), resp. \( x_M \), be any point of \( M \) such that \( \min_M \phi = \phi(x_m) \) and \( \max_M \phi = \phi(x_M) \). Denote by \(-A_{g_\phi} \), resp. \(-A_{g_\phi} \), the minimum of \( G_{g_\phi} \), resp. \( G_{g_\phi} \), on \( M \times M \), cf. [17, Theorem 4.13]. From (7.3.29)-(7.3.30) and (7.3.31)-(7.3.32), we get:

\[
\text{supphit} \quad (7.3.33) \quad \max_M \phi = \phi(x_M) = \frac{1}{V} \int_M \phi v_{g_0} + \int_M G_{g_\phi}(x_M, \cdot) \Delta_{g_\phi} \phi \, v_{g_0} \nonumber\]

\[
= \frac{1}{V} \int_M \phi v_{g_0} + \int_M (G_{g_\phi}(x_M, \cdot) + A_{g_\phi}) \Delta_{g_\phi} \phi \, v_{g_0} \nonumber
\]

\[
\leq \frac{1}{V} \int_M \phi v_{g_0} + m \int_M (G_{g_\phi}(x_M, \cdot) + A_{g_\phi}) v_{g_0} \nonumber
\]

\[
= \frac{1}{V} \int_M \phi v_{g_0} + m V A_{g_\phi},
\]

and

\[
-\min_M \phi = -\phi(x_m) = -\frac{1}{V} \int_M \phi v_{g_\phi} - \int_M G_{g_\phi}(x_m, \cdot) \Delta_{g_\phi} \phi \, v_{g_\phi} \nonumber
\]

\[
= -\frac{1}{V} \int_M \phi v_{g_\phi} - \int_M (G_{g_\phi}(x_m, \cdot) + A_{g_\phi}) \Delta_{g_\phi} \phi \, v_{g_\phi} \nonumber
\]

\[
\leq -\frac{1}{V} \int_M \phi v_{g_\phi} + m \int_M (G_{g_\phi}(x_m, \cdot) + A_{g_\phi}) v_{g_\phi} \nonumber
\]

\[
= -\frac{1}{V} \int_M \phi v_{g_\phi} + m V A_{g_\phi}.
\]
Suppose now that \( \{ \phi_t \}_{t \in [0,T)} \) is the maximal solution to (7.3.5) and that
\[
\frac{1}{V} \int_M \phi_t v_{g_{\phi_t}} \leq C,
\]
for some positive constant \( C \) independent of \( t \). From (7.3.33) and Proposition 4.2.2, we then infer:
\[
\max_M \phi_t \leq \frac{1}{V} \int_M \phi_t v_{g_0} + mVA_{g_0}
\]
\[
\leq \frac{m}{V} \int_M \phi_t v_{g_{\phi_t}} + mVA_{g_0}
\]
\[
\leq m(C + V\Omega A_{g_0}).
\]
From (7.3.34), we get
\[
\min_M \phi_t \leq \frac{1}{V} \int_M \phi_t v_{g_{\phi_t}} + mVA_{g_0}.
\]
For any \( \delta \) in \( (0,T) \), (7.3.18) implies \( r_{g_{\phi_t}} > k\delta g_{\phi_t} \) for any \( t \in [\delta,T) \). By Myers’ theorem — cf. e.g. [28, Theorem 6.51] — this implies \( \text{diam}_{g_{\phi_t}} < \frac{\sqrt{2m-1}}{\sqrt{k\delta}} \), where \( \text{diam}_{g_{\phi_t}} \) denotes the diameter of \( (M,g_{\phi_t}) \). This, in turn, implies
\[
VA_{g_{\phi_t}} < \gamma(m),
\]
for any \( t \) in \( [\delta,T) \), where \( \gamma \) only depends on \( m \), cf. [23, Theorem (3.2)]. We thus get
\[
- \min_M \phi_t \leq C + \frac{m\gamma(m)}{k\delta},
\]
for any \( t \) in \( [\delta,T) \), hence
\[
- \min_M \phi_t \leq \max_{s \in [0,\delta]} \left( - \min_M \phi_s \right) + C + \frac{m\gamma(m)}{k\delta},
\]
for any \( t \) in \( [0,T) \).

\section*{7.4. The Tian constant}

\textbf{Proposition 7.4.1 (G. Tian [181])}. Let \( (M,J) \) be a Fano manifold and fix the Kähler class \( \Omega = \frac{2\pi}{k} c_1(M,J) \), \( k > 0 \). Fix any base-point \( (g_0,\omega_0) \) in \( M_{\Omega} \). There then exists a positive real number \( \alpha \) and a positive real constant \( C_\alpha \), only depending on \( \alpha \) and on \( (M,J,\omega_0) \), such that
\[
\frac{1}{V} \int_M e^{2\alpha(\max_M \phi - \phi)} v_{g_0} \leq C_\alpha,
\]
for any \( \phi \) in \( \tilde{M}_{\omega_0} \).

\textbf{Proof}. Proposition 7.4.1 relies on the following general statement. In general a (smooth) real function \( f \) defined on a Kähler manifold \( (M,g,J,\omega) \) is called \textit{plurisubharmonic} if the \( J \)-invariant 2-form \( dd^c f \) is everywhere positive, meaning that the associated symmetric bilinear form \( X,Y \mapsto dd^c f(X,JY) \), which is the \( J \)-invariant part \( D^+\phi \) of the hessian of \( f \), up to a factor 2 — see Section 1.23 for the notation — is everywhere positive definite. By (1.15.6), we then have \( \Delta_g f \leq 0 \) everywhere.
LEMMA 7.4.1 ([104] Theorem 4.4.5). Let $a, b$ any two real number with $a < b$. For any positive real number $R$, denote by $B_R(0)$ the open ball of $\mathbb{C}^m$ centered at the origin $0$ and of radius $R$. Let $\mathcal{P}_{a,b,R}$ denote the set of those plurisubharmonic functions $\psi$ defined on $B_R(0)$ which satisfy the following two conditions:

$$(7.4.2) \quad \psi(0) \geq a, \quad \psi(\zeta) \leq b,$$

for any $\zeta$ in $B_R(0)$. Then, there exists a positive real number $C = C(a, b, R)$ such that

$$(7.4.3) \quad \int_{B_R(0)} e^{-\frac{\psi}{|x|^2}} v_{g_0} \leq C,$$

for any $\psi$ in $\mathcal{P}_{a,b,R}$.

PROOF. This is an easy generalization of Theorem 4.4.5 in L. Hörmander’s book [104].

In view of Lemma 7.4.1, the proof of Proposition 7.4.1 then goes as follows (we here closely follow unpublished notes kindly communicated by Y. Rubinstein). Let $\phi$ be any element of $\tilde{\mathcal{M}}_{\omega_0}$, so that $\omega = \omega_0 + dd^c \phi$ is the Kähler form of a Kähler metric $g_0$. Without loss of generality, we can assume $\max_M \phi = 0$. From (7.3.33) — which holds for any $\phi$ in $\tilde{\mathcal{M}}_{\omega_0}$ — we then infer

$$(7.4.4) \quad \int_M \phi \, v_{g_0} \geq -m A_{g_0} V_\Omega^2.$$ 

Since $\phi \leq 0$, for any open subset $U \subset M$ we infer $\sup_U \phi \geq \frac{1}{\Vol(U)} \int_U \phi \, v_{g_0} \geq \frac{1}{\Vol(U)} \int_M \phi \, v_{g_0} \geq -m A_{g_0} V_\Omega^2$, hence

$$(7.4.5) \quad \sup_U \phi \geq -\frac{m A_{g_0} V_\Omega^2}{\Vol(U)},$$

where $\Vol(U)$ denotes the volume of $U$ with respect to $g_0$. Since $M$ is compact, we may assume $M = \bigcup_{j=1}^N U_j$, for some positive integer $N$, where each $U_j$ is an open subset isomorphic to the open ball $B_0(R)$ in $\mathbb{C}^m$. We then set $U_j = B_{x_j}(R)$, where $x_j$ denotes the center of $U_j$ in the chosen identification $\tau_j : U_j \to B_0(R)$. We may moreover assume that the $U_j$ are chosen in such a way that $M = \bigcup_{j=1}^N B_{x_j}(R/6)$, with $B_{x_j}(R/6) = \tau^{-1}(B_0(R/6))$. In view of (7.4.5), in each $B_{x_j}(R/6)$, there exists $y_j$ such that

$$(7.4.6) \quad \phi(y_j) \geq -\frac{m A_{g_0} V_\Omega^2}{\Vol(B_{x_j}(R/6))}.$$ 

On the other hand, on each $U_j = B_{x_j}(R)$, $\omega_0$ admits a Kähler potential, say $\psi_j$, so that $\omega_0|_{U_j} = dd^c \psi_j$. By adding a constant to $\psi_j$, we may assume $\psi_j(y_j) = 0$. Moreover, each $\psi_j$ admits a (non-negative) upper bound on $B_{x_j}(5R/6)$ and we denote by $C_1$ the greatest of them, so that $\psi_j - C_1 \leq 0$ for any $j = 1, \ldots, N$. As a Kähler potential of $\omega$ on $U_j$, $\phi + \psi_j$ is pluriharmonic, as well as $\phi + \psi_j - C_1$, and we have

$$(7.4.7) \quad \phi + \psi_j - C_1 \leq 0.$$
on $B_{x_j}(5R/6)$, hence also on $B_{y_j}(2R/3) \subset B_{x_j}(5R/6)$, whereas, because of (7.4.6),

$$\text{(7.4.8)} \quad (\phi + \psi_j - C_2)(y_j) \geq -C_1 - \frac{mA_{y_0} V^2}{\text{Vol}(B_{x_j}(R/6))}. $$

By applying Lemma 7.4.1 to $\phi + \psi_j - C_2$ on the ball $B_{y_j}(2R/3) \subset U_j \cong B_0(R)$, for any $j = 1, \ldots, N$, we get $\int_{B_{y_j}(R/3)} e^{-a_j \phi} e^{-a_j (C_1 - \psi_j)} \frac{v_\alpha}{v_{y_0}} v_\gamma \leq C_j$, by setting $a_j = \left( C_1 + \frac{mA_{y_0} V^2}{\text{Vol}(B_{x_j}(R))} \right)^{-1}$, where $v_0$ denotes the volume form determined by the holomorphic chart $\tau_j : U_j \to B_0(R)$ and the constant $C_j$ is determined by $a_j$ and $R$; we infer $\int_M e^{-a_j \phi} v_\gamma \leq C_j$, where $C_j$ is the product of $C_j$ and the upper bound of $\frac{v_\alpha}{v_{y_0}}$ on $B_{y_j}(R/3)$. Finally, since $B_{x_j}(R/6) \subset B_{y_j}(2R/3)$, the open balls $B_{y_j}(2R/3)$ cover $M$ and we end up with

$$\text{(7.4.9)} \quad \int_M e^{-a \phi} v_\gamma \leq C,$$

with $a = \min_j a_j > 0$ and $C = \sum_{j=1}^N \tilde{C}_j$. By substituting $\phi - \max_M \phi$, we obtain (7.4.1) for any $\phi$ in $\mathcal{M}_\omega$. \hfill \Box

**Definition 10.** The Tian constant, $\alpha_M$, is defined as the upper bound of the set of those positive numbers $\alpha$ such that there exists a positive real number $C_\alpha$ so that (7.4.1) holds for any $\phi$ in $\mathcal{M}_\Omega$.

**Remark 7.4.1.** The Tian constant as defined above is independent of the chosen base-point $(\tilde{g}_0, \omega_0)$ in $\mathcal{M}_\Omega$. Indeed, if we substitute any other base-point $(\tilde{g}_0, \tilde{\omega}_0 = \omega_0 + dd^c \psi)$, and if (7.4.1) holds for some pair $\alpha, C$ and for any $\phi$ in $\mathcal{M}_\omega$, then (7.4.1) holds for the pair $\alpha, \tilde{C}$ and for any $\tilde{\phi} = \phi - \psi$ in $\mathcal{M}_{\tilde{\omega}_0}$, with, e.g. $\tilde{C}_\alpha = \sup_M \frac{v_\alpha}{v_{\tilde{\omega}_0}} e^{\alpha \text{sup}_M \psi - \text{inf}_M \psi}$. The Tian constant is also independent of the scaling $k$ of $\Omega$: if $\Omega = \frac{2\pi}{k} c_1(M, J)$ is replaced by $\Omega' = \frac{2\pi}{k} c_1(M, J)$ and $\omega_0 = \frac{k}{\tilde{k}} \omega_0$ is chosen as a base-point of $\mathcal{M}_\omega$, then, any $\phi'$ in $\mathcal{M}_{\omega_0}$ is of the form $\phi' = \frac{\tilde{k}}{k} \phi$, with $\phi$ in $\mathcal{M}_{\omega_0}$, so that $\int_M e^{k' \alpha \text{sup}_M \phi' - \text{inf}_M \phi'} v_\gamma = \left( \frac{\tilde{k}}{k} \right)^m \int_M e^{k \alpha \text{sup}_M \psi - \text{inf}_M \psi} v_\gamma$. The Tian constant $\alpha_M$ is then an invariant of the Fano manifold $(M, J)$.

**Theorem 7.4.1 (G. Tian [181]).** Let $(M, J)$ be a Fano manifold of Tian constant $\alpha_M$, equipped with the Kähler class $\Omega = \frac{2\pi}{k} c_1(M, J)$. Assume that

$$\text{(7.4.10)} \quad \alpha_M > \frac{m}{m + 1}. $$

Then, $\mathcal{M}_\Omega$ contains a Kähler-Einstein metric.

**Proof.** We first observe that the integral in (7.4.1) is invariant under the natural $\mathbb{R}$-action $a \cdot \phi = \phi + a$ on $\mathcal{M}_\Omega$: There is then no loss in restricting our attention to those $\phi$ in $\mathcal{M}_\Omega$ which belong to $L^1_{\omega_0}(0)$. Let $a$ be any positive number $\alpha$ satisfying (7.4.1) and let $\{\phi_t\}_{t \in [0, T]}$ be the maximal solution to the modified Monge-Ampère equation (7.3.5), as defined in Definition 9. From
(7.4.1) and (7.3.5), we infer:
\[
\log C_\alpha \geq \log \frac{1}{V_\Omega} \int_M e^{2k\alpha (\sup M \phi_t - \phi_0)} v_{g_0}
\]
(7.4.11)
\[
= \log \frac{1}{V_\Omega} \int_M e^{2k\alpha \sup M \phi_t + 2k(t - \alpha) \phi_t + p_{g_0} - c(t, \phi_t)} v_{g_{\phi_t}}
\]
\[
= 2k\alpha \sup M \phi_t + \log \frac{1}{V_\Omega} \int_M e^{2k(t - \alpha) \phi_t + p_{g_0} v_{g_{\phi_t}} - c(t, \phi_t)}.
\]
By using the Jensen inequality (7.3.13), we infer:
(7.4.12)
\[
\log C_\alpha \geq 2k\alpha \sup M \phi_t + \frac{1}{V} \int_M (2k(t - \alpha) \phi_t + p_{g_0}) v_{g_{\phi_t}} - c(t, \phi_t)
\]
\[
\geq 2k\alpha \frac{1}{V} \int_M \phi_t v_{g_0} + \frac{2k(t - \alpha)}{V} \int_M \phi_t v_{g_{\phi_t}} + \inf M p_{g_0} - c(t, \phi_t).
\]
By Proposition 4.2.2, \( \int_M \phi_t v_{g_0} \geq -\frac{1}{m} \int_M \phi_t v_{g_{\phi_t}} \geq 0 \), whereas, by Proposition 7.3.3, \( c(t, \phi_t) \leq c(0, \phi_0) \). The above inequality can then be rewritten as:
(7.4.13) \( \log C_\alpha \geq 2k \left( \frac{(m + 1)}{m} \alpha - t \right) - \frac{1}{V_\Omega} \int_M \phi_t v_{g_{\phi_t}} + \inf M p_{g_0} - c(0, \phi_0) \).
We observe that the coefficient in front of \( -\frac{1}{V_\Omega} \int_M \phi_t v_{g_{\phi_t}} \) in (7.4.13) is positive for each \( t \) in \([0, 1]\) if and only if \( \alpha \) is greater than \( \frac{m}{m + 1} \) and that \( \alpha \) can be chosen that way if and only if the Tian constant \( \alpha_M \) satisfies (7.4.10). If so, we get
(7.4.14) \( \frac{1}{V_\Omega} \int_M \phi_t v_{g_{\phi_t}} \leq \frac{\log C_\alpha - \inf M p_{g_0} + c(0, \phi_0)}{2k \left( \frac{(m + 1)}{m} \alpha - 1 \right)} \)
and we conclude by using Proposition 7.3.5. \( \square \)

Remark 7.4.2. In the case when \( \text{Aut}(M, J) \) is not reduced to the identity, Theorem 7.4.1 can be significantly improved by considering any compact subgroup — possibly non connected — of \( \text{Aut}(M, J) \), and by substituting the Tian constant relative to \( G, \alpha_M^G \), defined a the upper bound of the set of positive real numbers \( \alpha \) such that (7.4.1) holds for any \( \phi \) in \( \mathcal{M}_\omega^G \), the space of \( G \)-invariant relative Kähler potentials with respect to some arbitrary \( G \)-invariant \( \omega_0 \) in \( \mathcal{M}_\Omega \).

7.5. The Ding functional

The Ding functional, \( F_{\omega_0} \), was first introduced on Fano manifolds by Wei-Yue Ding in [68], as a functional on \( \mathcal{M}_\Omega \) — \( \Omega = \frac{2\pi}{k} c_1(M, J), k > 0 \) — defined by
(7.5.1) \( F_{\omega_0}(\omega_\phi) = 2k \left( J_{\omega_0}(\omega_\phi) - \frac{1}{V} \int_M \phi v_{g_0} \right) - \log \frac{\int_M e^{-p_{g_0} - 2k\phi} v_{g_{\phi_0}}}{\int_M e^{-p_{g_0}} v_{g_0}} \),
for any \( \omega_\phi = \omega_0 + dd^c \phi \) in \( \mathcal{M}_\Omega \) (notice that this expression is independent of the chosen normalization of the Ricci potential \( p_{g_0} \)). By using (4.2.7), this
7.5. The Ding Functional

The Ding functional can be re-written as

\[ F_{\omega_0}(\omega) = -2k \mathbb{I}_{\omega_0}(\phi) - \log \int_M e^{-p_{\omega_0} - 2k\phi} v_{g_0}. \]

It follows that \( F_{\omega_0} \) can be regarded as the restriction of \(-2k \mathbb{I}_{\omega_0}\) to the space of relative Kähler potentials \( \phi \) in \( \mathcal{M}_{\omega_0} \) normalized by the condition

\[ \int_M e^{-p_{\omega_0} - 2k\phi} v_{g_0} = \int_M e^{-p_{\omega_0}} v_{g_0}. \]

In the setting of this Chapter, where the relative Kähler potential \( \phi \) is normalized by \( \mathbb{I}_{\omega_0}(\phi) = 0 \), \( F_{\omega_0} \) has then the following alternative expression:

\[ F_{\omega_0}(\omega) = -\log \frac{1}{V} \int_M e^{-p_{\omega_0} - 2k\phi} v_{g_0} + \log \frac{1}{V} \int_M e^{-p_{\omega_0}} v_{g_0}. \]

**Proposition 7.5.1.** The derivative of the Ding functional \( F_{\omega_0} \) at \((g,\omega)\) in \( \mathcal{M}_\Omega \) is given by

\[ (dF_{\omega_0})_g(f) = 2k \int_M e^{-p_g} f v_g \]

for any \((g,\omega = \omega_0 + dd^c \phi)\) in \( \mathcal{M}_\Omega \) — identified with \( \mathbb{I}_{\omega_0}^{-1}(0) \) in \( \tilde{\mathcal{M}}_{\omega_0} \) — and any \( f \) in \( T_\omega \mathcal{M}_\Omega \), with \( \int_M f v_g = 0 \). In particular, \( \omega \) is critical for \( F_{\omega_0} \) if and only if \( p_g \equiv 0 \), hence if and only if \((g,\omega)\) is Kähler-Einstein.

**Proof.** We readily deduce from (7.5.4) that

\[ (dF_{\omega_0})_g(f) = 2k \int_M e^{-p_{\omega_0} - 2k\phi} f v_{g_0}. \]

We then deduce (7.5.5) by using (7.1.4). From (7.5.5), we readily infer that \((g,\omega)\) is critical for \( F_{\omega_0} \) if and only if \( p_g \equiv 0 \), hence identically zero. By the very definition (1.19.10)-(1.19.11) of the Ricci potential \( p_g \), this means that \((g,\omega)\) is Kähler-Einstein. \( \square \)

It follows from (7.5.5) that the Ding functional \( F_{\omega_0} \) is the primitive vanishing at \( \omega_0 \) of the 1-form \( \varpi \) on \( \mathcal{M}_\Omega \) defined by

\[ \varpi(g) = 2k \frac{e^{-p_g} v_g}{\int_M e^{-p_g} v_g}, \]

riemannian dual of the vector field \( \hat{p} \) defined by

\[ \hat{p}(g) = 2k \frac{e^{-p_g}}{\int_M e^{-p_g} v_g} - 2k, \]

in the same way as the Mabuchi K-energy was defined as the primitive vanishing at \( \omega_0 \) of the 1-form \( \sigma(g) = s_g v_g \), cf. Section 4.10, in particular Lemma 4.10.1, for the notation.

**Lemma 7.5.1.** The covariant derivatives of \( \hat{p} \) and \( \varpi \) relative to Mabuchi connection \( \mathcal{D} \) are given by

\[ \mathcal{D}_f \hat{p} = 2k \frac{(\mathcal{D}_f g - 2k) \int e^{-p_g}}{\int_M e^{-p_g} v_g}, \]

\( \hat{p} \) and \( \varpi \) being defined as above. \( \square \)
and

\[(7.5.10) \quad D_f \varpi = 2k \frac{(\mathbb{D}_g - 2k) \tilde{f} e^{-p_g} v_g}{\int_M e^{-p_g} v_g},\]

for any \( g \) in \( \mathcal{M}_\Omega \) and any \( f \) in \( T_g \mathcal{M}_\Omega = \{ f \in C^\infty(M, \mathbb{R}) | \int_M f v_g = 0 \} \), where \( \tilde{f} \) is defined by

\[(7.5.11) \quad \tilde{f} = f - \frac{\int_M f e^{-p_g} v_g}{\int_M e^{-p_g} v_g}.

**Remark**

**Proposition** (7.5.12) \( (7.5.13) \), the latter is given by

\[(7.5.12) \quad D_f \hat{\rho} = f \cdot \hat{\rho} - (d\hat{\rho}(g), df)_g,

where \( f \cdot \hat{\rho}(g) \) stands for the first variation of \( \hat{\rho} \) along \( f \). By (7.1.11) and (3.2.4), the latter is given by

\[(7.5.13) \quad f \cdot \hat{\rho} = -\frac{2k e^{-p_g}}{\int_M e^{-p_g} v_g} \left[ -\Delta_g f + 2k f + \frac{1}{V_M} \int_M p_g \Delta_g f v_g \right] + \frac{2k e^{-p_g}}{\left( \int_M e^{-p_g} v_g \right)^2} \int_M \left( 2k f + \frac{1}{V_M} \int_M p_g \Delta_g f v_g \right) e^{-p_g} v_g

whereas

\[(7.5.14) \quad -(d\hat{\rho}(g), df)_g = 2k \frac{(dp_g, df) e^{-p_g}}{\int_M e^{-p_g} v_g}.

By putting (7.5.13)–(7.5.14) together we get (7.5.9). \( \square \)

**Proposition 7.5.2.** The Ding functional \( F_{\omega_0} \) is convex with respect to the Mabuchi connection \( D \), i.e. its hessian with respect to \( D \) is semi-positive on \( \mathcal{M}_\Omega \). More precisely,

\[(7.5.15) \quad (D_f dF_{\omega_0})(f) \geq 0,

for any \( g \) in \( \mathcal{M}_\Omega \) and any \( f \) in \( T_g \mathcal{M}_\Omega \), with equality at \( g \) if and only if \( f \) is a Killing potential with respect to \( g \).

**Proof.** Since \( \varpi = d F_{\omega_0} \), from (7.5.10) we infer that \( (D_f dF_{\omega_0})(f) = 2k \frac{\int_M (\mathbb{D}_g - 2k) \tilde{f} e^{-p_g} v_g}{\int_M e^{-p_g} v_g} \). We then conclude by using Proposition 7.2.2. \( \square \)

**Remark 7.5.1.** Proposition 7.5.2 implies that the Ding functional is convex along each geodesic of \( \mathcal{M}_\Omega \), as is the Mabuchi K-energy, cf. Sections 4.10 and 4.15. The convexity of the Ding functional along (weak) geodesics of \( \mathcal{M}_\Omega \) was first established by B. Berndtsson in [27].

**Proposition 7.5.3 ([70] Formula (3.8), [170] Lemma 2.3).** The Ding functional is related to the Mabuchi K-energy by

\[(7.5.16) \quad F_{\omega_0}(\omega_0) = \frac{1}{V} E_{\omega_0}(\omega_0) - \log \frac{1}{V} \int_M e^{-p_{\omega_0} v_{\omega_0}} + \log \frac{1}{V} \int_M e^{-p_{\omega_0} v_{\omega_0}},\]
for any \((g_0, \omega_\phi = \omega_0 + d\bar{\phi})\) in \(\mathcal{M}_\Omega\). In particular, \(F_{\omega_0}\) is bounded from above by \(E_{\omega_0}\) in the following sense:

\[
F_{\omega_0}(\omega) \leq \frac{1}{V} E_{\omega_0}(\omega) + \log \frac{1}{V} \int_M e^{-p_{g_0} v_{g_0}}.
\]

**Proof.** By integrating \(e^{p_\phi}\) along \(v_{g_\phi}\), when the Ricci potential \(p_{g_\phi}\) is given by \((7.1.4)\), we easily get the following expression of the \(K\)-energy on \(\mathcal{M}_\Omega = \mathbb{L}^{-1}(0)\):

\[
\frac{1}{V} E_{\omega_0}(\omega_\phi) = - \log \frac{1}{V} \int_M e^{-p_{g_0} - 2k_{\phi} v_{g_0}} + \log \frac{1}{V} \int_M e^{-p_{g_\phi} v_{g_\phi}}.
\]

By comparing with \((7.5.4)\), we readily get \((7.5.16)\). We then deduce \((7.5.17)\) by using the Jensen inequality \((7.3.13)\):

\[
\log \frac{1}{V} \int_M e^{-p_{g_\phi} v_{g_\phi}} \geq - \frac{1}{V} \int_M p_{g_0} v_{g_0} \geq 0.
\]

The following definition is due to G. Tian, cf. Definition 4.5 in [185]:

**Definition 11.** The Ding functional is said to be \textit{proper with respect to the Aubin functional} \(I_{\omega_0}\), or simply \textit{proper}, if there exists an increasing function \(h = h(s)\) defined on \(\mathbb{R}\), with \(\lim_{s \to +\infty} h(s) = +\infty\), such that

\[
F_{\omega_0}(\omega_\phi) \geq h(I_{\omega_0}(\omega_{\phi_i})),
\]

for any \(\omega_\phi = \omega_0 + d\bar{\phi}\) in \(\mathcal{M}_\Omega\).

**Remark 7.5.2.** It is easy to check that the properness of \(F_{\omega_0}\) as defined in \((7.5.19)\) is independent of the chosen base-point \(\omega_0\) in \(\mathcal{M}_\Omega\).

**Proposition 7.5.4 (G. Tian).** If the Ding functional \(F_{\omega_0}\) is proper, the maximal solution \(\{\phi_t\}_{t \in [0, T)}\) smoothly converges to the relative Kähler potential of a Kähler-Einstein metric.

**Proof.** Assume that \(F_{\omega_0}\) is proper and consider \(F_{\omega_0}(\omega_{g_\psi_t})\), where \(\{\psi_t\}_{t \in [0, T)}\) is the maximal solution to the deformed Monge-Ampère equation \((7.3.5)\) relative to any base-point \(\omega_0\) in \(\mathcal{M}_\Omega\), as defined in Definition 9 (in particular, \(I_{\omega_0}(\psi_t) = 0\) for any \(t \in [0, T)\)). By using \((7.5.17)\), \((7.3.26)\) and \((7.5.19)\), we thus get

\[
h(I_{\omega_0}(\omega_{\phi_t})) \leq F_{\omega_0}(\omega_{\phi_t}) \leq -c(0) + \log \frac{1}{V} \int_M e^{-p_{g_0} v_{g_0}}.
\]

We infer that \(I_{\omega_0}(\omega_{\phi_t})\) is uniformly bounded from above and we conclude that \(\omega_{\phi_t}\) smoothly converges to a Kähler-Einstein metric by using Proposition 7.3.5, together with Proposition 4.2.2 and (4.2.5). \(\square\)
### Polarized Kähler manifolds

#### 8.1. Polarized Kähler manifolds

In general, a symplectic manifold \((M, \omega)\) is said to be (pre)quantized by a hermitian complex line bundle \((L, h)\) if \(\omega\) can be realised as the curvature form of some hermitian connection \(\nabla\) of \(L\), so that

\[
R^\nabla = i\omega,
\]

(8.1.1)

cf. Section 1.19.

**Proposition 8.1.1.** A symplectic manifold \((M, \omega)\) can be (pre)quantized as above if and only if the de Rham class \([\frac{\omega}{2\pi}]\) is integral, i.e. belongs to the image of \(H^2(M, \mathbb{Z})\) in \(H^2_{\text{dR}}(M, \mathbb{R})\).

**Proof.** A detailed proof of this basic and well-known fact would go beyond the scope of these notes and we only give a sketchy argument. Recall that for any hermitian connection \(\nabla\) on \(L\), the real 2-form \(R^\nabla\) is closed and represents the (real) Chern class \(c^R(L)\) of \(L\) in \(H^2(M, \mathbb{R})\) (\(c^R(L)\) is actually represented by \(R^\nabla\) for any \(\mathbb{C}\)-linear connection \(\nabla\) on \(L\), but we here only consider hermitian connections). It turns out that \(c^R(L)\) is the image in \(H^2_{\text{dR}}(M, \mathbb{R})\) of an element of \(H^2(M, \mathbb{Z})\), called the (integral) Chern class of \(L\), denoted by \(c(L)\), and that the map \(L \mapsto c(L)\) determines a group isomorphism of \(H^2(M, \mathbb{Z})\) with the set of classes of isomorphisms of complex line bundles over \(M\), equipped with the abelian group structure induced by the (complex) tensor product (we then have: \(c(L_1 \otimes L_2) = c(L_1) + c(L_2)\)), cf. e.g. [100] Chapter 1. It follows that a necessary condition that \((M, \omega)\) be (pre)quantized by some complex line bundle if that \([\frac{\omega}{2\pi}]\) be integral, as defined above (equivalently, the periods \(\int_C \omega\) of \(\omega\) on all closed 2-cycles \(C\) of \(M\) are integers). Conversely, if \([\frac{\omega}{2\pi}]\) is integral, it is the real Chern class of a complex line bundle \(L\), well-defined up to a torsion element in \(H^2(M, \mathbb{Z})\). For any hermitian inner product \(h\) and any \(h\)-hermitian connection \(\nabla\) on \(L\), \(-iR^\nabla\) and \(\omega\) then belong to the same de Rham class, so that \(R^\nabla = i(\omega + d\alpha)\), for some real 1-form \(\alpha\). By replacing \(\nabla\) by \(\nabla + i\alpha\), which is again a hermitian connection on \(L\), we get \(R^\nabla + i\alpha = R^\nabla - id\alpha = i\omega\). \(\square\)

We now consider the case when the symplectic form \(\omega\) is the Kähler form of a Kähler structure \((g, J, \omega)\) on \(M\). If \(\omega\) is (pre)quantized as above, then \(\nabla\) is the Chern connection of a uniquely defined holomorphic structure on \(L\), as its curvature \(R^\nabla = i\omega\) is \(J\)-invariant, cf. Proposition 1.6.1.

We accordingly change our point of view and say that a Kähler manifold \((M, g, J, \omega)\) is polarized by a holomorphic line bundle \(L\) if \(L\) admits a hermitian inner product \(h\) such that the Kähler form \(\omega\) is equal to the curvature form of the corresponding Chern connection, cf. Section 1.7.
then say that the Kähler manifold is polarized by the hermitian holomorphic line bundle \((L, h)\) and the Kähler form will be occasionally written \(\omega_h\).

By Proposition 8.1.1, a Kähler manifold \((M, g, J, \omega)\) can be polarized by a holomorphic line bundle \(L\) if and only if \([\omega^2]\) is integral in \(H^2_{\text{dR}}(M, \mathbb{R})\).

**Definition 12.** Let \((M, J)\) be a complex manifold and \(L\) a holomorphic line bundle over \(M\). Then, \(L\) is said to be ample if there exists a Kähler metric \((g, J, \omega)\) on \(M\) which is polarized by \(L\). Equivalently, \(L\) is ample if there exists a hermitian inner product \(h\) on \(L\) such that the curvature form of the corresponding Chern connection is positive, i.e., the Kähler form of a Kähler metric on \((M, J)\). The holomorphic hermitian line bundle \((L, h)\) is then called positive.

Let \((M, g, J, \omega)\) be a Kähler manifold polarized by a holomorphic line bundle \(L\) and denote by \(h\) a hermitian inner product on \(L\), unique up to rescaling, such that the Kähler form \(\omega\) be the curvature form of the induced Chern connection. By Proposition 1.7.1, any (local) non-vanishing holomorphic section \(s\) of \(L\) provides a (local) Kähler potential, by

\[
\omega = \frac{1}{2} \dd c \log |s|^2_h,
\]

where \(|s|^2_h\) denotes the square norm of \(s\) with respect to \(h\). If \(\tilde{L}\) denotes the corresponding \(\mathbb{C}^*\)-principal bundle, regarded as as the bundle of non-zero elements of \(L\), and \(\tilde{\pi}\) the natural (holomorphic) projection from \(\tilde{L}\) to \(M\), we then have that

\[
\tilde{\pi}^* \omega = -\dd c \log r_h,
\]

where \(r_h\) stands for the restriction to \(\tilde{L}\) of the norm function on \(L\) determined by \(h\). For any polarized Kähler manifold \((M, L, h)\), the function \(-\log r_h\) on \(\tilde{L}\) then appears as a substitute of a (in general missing) globally defined Kähler potential.

**Remark 8.1.1.** According to (6.2.3), the standard complex projective space, equipped with the Fubini-Study Kähler structure of holomorphic sectional curvature \(c = 2\) defined in Section 6.2, satisfies

\[
\rho^{\Lambda^*} = \omega_{FS},
\]

and is then polarized by the dual tautological bundle \(\Lambda^* = \mathcal{O}(1)\). From now on, unless otherwise specified, the term *Fubini-Study metric* will implicitly mean *Fubini-Study metric of holomorphic sectional curvature +2*, i.e., the Kähler metric polarized by the dual tautological bundle for some hermitian inner product.

Any (compact) complex submanifold, \(M\), of \(\mathbb{P}^m = \mathbb{P}(\mathbb{C}^{m+1})\) inherits a Kähler structure, whose metric and Kähler form are the restrictions of \(g_{FS}\) and \(\omega_{FS}\) to \(M\) (note that \(\omega_M := \omega_{FS}|_M\) is closed again, so that the induced almost hermitian structure is still Kähler by Proposition 1.1.1). This metric is then polarized by the restriction to \(M\) of the dual tautological line bundle \(\Lambda^*\) to \(M\), equipped with the hermitian inner product induced by the standard hermitian inner product of \(\mathbb{C}^{m+1}\).

A *projective manifold* is a compact complex manifold that can be realized as a complex submanifold of some complex projective. Any holomorphic
embedding of a projective manifold in a complex projective space then turns it into a polarized Kähler manifold. Conversely, any compact polarized Kähler manifold is a projective manifold. This is the main content of the celebrated Kodaira embedding theorem, of which a precise statement is given in Section 8.3.

We close this section by giving an alternative description of the reduced automorphism group $H_{\text{red}}(M, J)$ defined in Section 2.4 for a polarized compact Kähler manifold $(M, L)$. Let $L$ be a holomorphic line bundle over a compact complex manifold $(M, L)$ and denote by $\pi$ the natural (holomorphic) projection from $L$ to $M$. By an automorphism of $L$ we mean a biholomorphic map, $\tilde{\Phi}$, from $L$ to itself such that there exists an element $\Phi$ of $\text{Aut}(M, J)$ and a commutative diagram

$$
\begin{array}{ccc}
L & \xrightarrow{\tilde{\Phi}} & L \\
\downarrow \pi & & \downarrow \pi \\
M & \xrightarrow{\Phi} & M
\end{array}
$$

with the following property: for any $x$ in $M$, the restriction of $\tilde{\Phi}$ to $\pi^{-1}(x)$ is a $\mathbb{C}$-linear isomorphism from $\pi^{-1}(x)$ to $\pi^{-1}(\Phi(x))$. The automorphisms of $L$ form a group denoted by $\text{Aut}(L)$; the identity component of $\text{Aut}(L)$ is denoted by $H(L)$. Two elements $\Phi, \Phi'$ of $\text{Aut}(L)$ induce the same $\Phi$ in $\text{Aut}(M, J)$ if and only if there exists an invertible holomorphic section, say $\zeta$, of the holomorphic vector bundle $\text{End}(L) = L^* \otimes L$ such that $\Phi' = \zeta \circ \Phi$.

Since $L$ is of rank 1, we have $\text{End}L = M \times \mathbb{C}$, so that $\zeta$ can be viewed as a non-vanishing holomorphic complex function, hence a (non-zero) constant number, as $M$ is compact. We then have the following exact sequences of group homomorphisms:

$$
\begin{array}{c}
1 \longrightarrow \mathbb{C}^* \longrightarrow \text{Aut}(L) \longrightarrow \text{Aut}_L(M, J) \longrightarrow 1 \\
\uparrow \quad \quad \quad \uparrow \\
1 \longrightarrow \mathbb{C}^* \longrightarrow H(L) \longrightarrow H_L(M, J) \longrightarrow 1
\end{array}
$$

where $\text{Aut}_L(M, J)$, resp. $H_L(M, J)$, denotes the group of those elements of $\text{Aut}(M, J)$, resp. $H(M, J)$, which lift to $\text{Aut}(L)$, resp. to $H(L)$. By definition, $H_L(M, J)$ is a subgroup of $H(M, J)$ and its Lie algebra, call it $\mathfrak{h}_L(M, J) = \mathfrak{h}_L$, is then a Lie subalgebra of $\mathfrak{h}(M, J) = \mathfrak{h}$. It turns out that $H_L(M, J), \mathfrak{h}_L(M, J)$, are actually independent of $L$. More precisely, we have (cf. [30] lemme 1.1):

**Proposition 8.1.2.** Let $(M, g, J, \omega)$ be a compact Kähler manifold and suppose that it is polarized by some holomorphic line bundle $L$. Then,

$$(8.1.7) \quad H_L(M, J) = H_{\text{red}}(M, J);$$

equivalently,

$$(8.1.8) \quad \mathfrak{h}_{\text{red}} = \mathfrak{h}_L.$$

**Proof.** From the above, we infer that $H_L(M, J)$ can be viewed as the quotient of $H(L)$ by the natural action of $\mathbb{C}^*$. In view of Theorem 3.1 in
the Lie algebra of $H(L)$ is the space of those complete $\mathbb{C}^*$-invariant (real) holomorphic vector fields defined on the total space of $L$, which project on a (real) holomorphic vector field of $M$, and $\mathfrak{h}_L$ is then the quotient of this Lie algebra by the space of vector fields generated by the natural action of $\mathbb{C}^*$ on $L$, namely the space of vector fields of the form $aT + biT$, for any two real numbers $a, b$, where $T$ denotes the (real) vector field on $L$ generated by the natural action of $S^1$, so that $T(u) = iu$ for any $u$ in $L$.

Let $X$ be any element of $\mathfrak{h}_L \subset \mathfrak{h}$ and denote by $\hat{X}$ any lift of $X$ in the Lie algebra of the group of automorphisms of $L$ ($\hat{X}$ is then well-defined up to vector fields of the form $aT + biT$ as above). Denote by $\nabla$ the Chern connection on $L$ such that $R_{\nabla} = i\omega$ and by $H_{\nabla}$ be the associated horizontal distribution on the total space of $L$, cf. Section 1.6. Since $\hat{X}$ projects to $X$, it can be written as

\begin{equation}
\hat{X} = \tilde{X} - hT + f iT,
\end{equation}

where $\tilde{X}$ denotes the horizontal lift of $X$ with respect to $H_{\nabla}$ and $f, h$ are real functions, defined up to additive constants. Since $\tilde{X}$ is $\mathbb{C}^*$-invariant, and $\tilde{X}$ is already $\mathbb{C}^*$-invariant — as $\nabla$ is a $\mathbb{C}$-linear connection — $f, h$ are actually constant on each fiber: they can therefore be considered as real functions defined on $M$ and we then normalize them by $\int_M f \omega^m = \int_M h \omega^m = 0$. It remains to express the condition that $L_{\hat{X}} \mathcal{J} = 0$, where $\mathcal{J}$ stands for the complex structure operator defined on the total space of $L$. In order to compute $L_{\hat{X}} \mathcal{J}$, it is convenient to use auxiliary vector fields, namely horizontal lifts, $\tilde{Y}$, of vector fields $Y$ on $M$ and sections, $s$, of $L$, viewed as vertical vector fields on $L$, constant on each fibre, cf. Section 1.6. As already mentioned, cf. (1.6.11), the brackets of these vector fields on the total space of $L$ are given by

\begin{equation}
[\tilde{Y}, s] = \nabla_Y s,
\end{equation}

whereas their brackets with $T, iT$ are given by

\begin{equation}
[T, \tilde{Y}] = [iT, \tilde{Y}] = 0,
\end{equation}

as $\nabla$ is a $\mathbb{C}$-linear connection, and by

\begin{equation}
[T, s] = -is, \quad [iT, s] = s,
\end{equation}

as $T(u) = iw$ and $(iT)(u) = -u$, whereas, we recall, $s$, as a vertical vector field, is constant on each fibre. We also recall — cf. (1.6.12) in Section 1.6 — that for any two vector fields $X, Y$ on $M$, the bracket $[\tilde{X}, \tilde{Y}]$ is the sum of a horizontal vector field, namely the horizontal lift $[X, Y]$ of the bracket $[X, Y]$, and of a vertical vector field, namely the linear vertical vector field $R_{X, Y} = \omega(X, Y)T$. On the other hand, for any $u$ in $L$, with $\pi(u) = x$, we have that: $T_uL = H_u^\nabla \oplus T_u^M \cong T_xM \oplus L_x$ and $\mathcal{J}$ is then $J$ on $T_xM$ and $i$ on $L_x$ (cf. Section 1.6). We are now ready to compute $L_{\tilde{X}} \mathcal{J}$. Since
\[ df(T) = df(iT) = dh(T) = dh(iT) = 0, \]

where \( s \) is any section \( L \). Then,

\[
(L_XJ)(\tilde{Y}) = [\tilde{X}, J\tilde{Y}] - J[\tilde{X}, \tilde{Y}]
\]

\[
= [\tilde{X}, J\tilde{Y}] - J[\tilde{X}, \tilde{Y}]
+ dh(JY)^T - df(JY)iT - dh(Y)iT - df(Y)^T
\]

\[(8.1.14)\]

for any vector field \( Y \) on \( M \). This part of \( L_XJ \) vanishes if and only if \( X = \text{grad} f + J\text{grad} h \), if and only if \( X \) belongs to \( h_{\text{red}} \). It follows that \( h_L \) is contained in \( h_{\text{red}} \).

Conversely, for any \( X = \text{grad} f + J\text{grad} h \) in \( h_{\text{red}} \), the vector field \( \tilde{X} := \tilde{X} - hT + f iT \) is a \( \mathbb{C}^* \)-invariant, holomorphic vector field on \( L \), which projects on \( X \): it then belongs to \( h_L \), provided that its flow is complete. In order to check that \( \tilde{X} \) is complete for any \( X \) in \( h_{\text{red}} \), we use the following completeness criterion taken from [1]:

**Proposition 8.1.3** ([1] Proposition 2.1.20). Let \( Z \) be a (smooth) vector field, \( k \geq 1 \), defined on some manifold \( N \), and let \( \phi \) a (smooth) real proper function defined on \( N \). Suppose that there exist two non-negative constants \( A, B \) such that

\[(8.1.15)\]

\[ |d\phi(Z)| \leq A|\phi| + B. \]

Then, the flow of \( Z \) is complete.

For a proof of this criterion, the reader is referred to [1], where the argument is given for the more general situation when \( Z \) is \( C^k \), \( k \geq 1 \), and \( \phi \) is \( C^1 \). In the current situation, we can choose \( \phi = r^2 \) on \( L \) — the square norm function \( r^2 \) determined by \( h \) — which is proper as \( M \) is compact. We then have \( d\phi(\tilde{X}) = -2fr^2 = -2f\phi \) and (8.1.15) is then evidently satisfied with \( A = 2\text{Sup}_M|f| \) and \( B = 0 \). This completes the proof of Proposition 8.1.2.

**Remark 8.1.2.** For any compact Kähler manifold \((M, J, g_0, \omega_0)\) polarized by an ample hermitian holomorphic line bundle \((L, h_0)\), the space \( \mathcal{M}_{\omega_0} \) defined in Section 4.1 can be identified with the space, \( \mathcal{H}^2(L) \), of hermitian inner products on \( L \) such that the curvature form of the corresponding Chern connection is positive, i.e. the Kähler form of a Kähler structure in \( \mathcal{M}_\Omega \), with \( \Omega = 2\pi c_1(L) \). The identification goes as follows, cf. [74]. Any hermitian inner product \( h \) on \( L \) is conformal to \( h_0 \), as \( L \) is of (complex) rank
1, hence of the form $h = e^{-2\phi} h_0$, for some real function $\phi$. By Proposition 1.7.2, the Chern curvature form of $h$, say $\omega$, is then given by

$$(8.1.16) \quad \omega = \omega_0 + dd^c \phi.$$ 

It follows that $h$ belongs to $\mathcal{H}^{>0}(L)$ if and only if $\phi$ belongs to $\hat{\mathcal{M}}_{\omega_0}$. The obvious advantage of considering $\mathcal{H}^{>0}(L)$, instead of $\hat{\mathcal{M}}_{\omega_0}$, is that its definition is independent of the choice of $\omega_0$ in $\mathcal{M}_\Omega$, as well as the natural projection, $p$, say, from $\mathcal{H}^{>0}(L)$ to $\mathcal{M}_\Omega$, which is simply the map $h \mapsto \rho^V$, where, we recall, $\rho^V$ denotes the corresponding Chern curvature form. Again, $p$ makes $\mathcal{H}^{>0}(L)$ into a $\mathbb{R}$-principal bundle over $\mathcal{M}_\Omega$, via the $\mathbb{R}$-action on $\mathcal{H}^{>0}(L)$ defined by $a \cdot h = e^{-2a} h$, for any real number $a$. The whole structure of $\hat{\mathcal{M}}_{\omega_0}$, as described in Section 4.1, can then be transported on $\mathcal{H}^{>0}(L)$, in particular the connection 1-form $\tau$ and its primitive $\|_\omega$, cf. Section 8.7 below.

### 8.2. The Bergman density of a polarized Kähler manifold

In this section $(M, g, J, \omega)$ denotes a (connected) compact Kähler manifold polarized by a hermitian holomorphic line bundle $(L, h)$ in the sense of (8.1.1). The (pointwise) hermitian inner product $h$ on $L$ together with the volume form $v_g$ of $g$ then determine a hermitian inner product, $\langle \cdot, \cdot \rangle_h$, on the space $\Gamma(L)$ of (smooth) sections of $L$ defined by

$$(8.2.1) \quad (s, \bar{s})_h = \int_M \langle s, \bar{s} \rangle_h v_g,$$

for any two $s, \bar{s}$ in $\Gamma(L)$. By restriction, this induces a hermitian inner product on the (finite dimensional) space $H^0(M, L)$ of holomorphic sections of $L$, still denoted by $\langle \cdot, \cdot \rangle_h$.

**Definition 13.** For any compact Kähler manifold $(M, g, J, \omega)$ polarized by a hermitian holomorphic line bundle $(L, h)$, the **Bergman density**, $B_{L,h}$, is the function defined on $M$ by

$$(8.2.2) \quad B_{L,h}(x) := \sum_{j=0}^N |s_j(x)|^2_h,$$

for any orthonormal basis $s = \{s_0, s_1, \ldots, s_N\}$ of $H^0(M, L)$ with respect to the (global) hermitian inner product $\langle \cdot, \cdot \rangle_h$.

**Remark 8.2.1.** The Sawyer Bergman operator, $\Pi_L$, is defined as the orthogonal projector from the Hilbert space $L^2(M, L)$ of $L^2$-sections of $(L, \langle \cdot, \cdot \rangle)$ to the (finite dimensional) subspace $H^0(M, L)$; for any $s$ in $L^2(M, L)$ we then have $\Pi_L(s) = \sum_{j=0}^N \langle s, s_j \rangle_h s_j$ for any orthonormal basis $s = \{s_0, \ldots, s_N\}$ of $H^0(M, L)$. The **Bergman kernel** is then defined by $B_L(x, y) = \sum_{j=0}^N s_j(y) \otimes \bar{s}_j(x)$ in $L_y \otimes L_x^*$, for any pair $(x, y)$ in $M \times M$ and $B_{L,h}$ can then regarded as the restriction of the Bergman kernel to the diagonal $M \times M$ (for more information on the Bergman kernel we refer the reader to [140] and references therein).

The following theorem concerns the behaviour of the Bergman density $B_{L^k,h}^{(k)}$ when the positive integer $k$ tends to infinity. Here, $B_{L^k,h}^{(k)}$ is
the Bergman density of the polarized Kähler manifold \((M, L^k)\) when \(L^k\) is equipped with the hermitian inner product \(h^{(k)}\) induced by \(h\), so that:

\[
B_{L^k, h^{(k)}}(x) = \sum_{i=0}^{N_k} |s_i^{(k)}(x)|^2_{h^{(k)}},
\]

for any orthonormal basis \(s^{(k)}\) of \(H^0(M, L^k)\) with respect to the hermitian inner product \(\langle \cdot, \cdot \rangle_{h^{(k)}}\) of \(H^0(M, L^k)\) determined by \(h^{(k)}\) and the volume form \(v_g = k^m v_g\). We then have:

**Theorem 8.2.1** (S. Zelditch [200], D. Catlin [52], Z. Lu [138]). For any compact Kähler manifold \((M, g, J, \omega)\) of complex dimension \(m\), polarized by a hermitian holomorphic line bundle \((L, h)\), the Bergman density \(B_{L^k, h^{(k)}}\) of \((M, L^k)\) admits an asymptotic expansion

\[
B_{L^k, h^{(k)}} \simeq \frac{1}{(2\pi)^m} \sum_{r=0}^{+\infty} \frac{\phi_r}{k^r},
\]

when \(k\) tends to infinity, where, for each non-negative integer \(r\), \(\phi_r\) is a smooth function on \(M\) which is a universal polynomial expression of the curvature \(R\) of \(M\) and the covariant derivatives of \(R\) up to the order \(2r - 1\).

The asymptotic expansion is understood in the following sense: for any non-negative integers \(q, \ell\), there exists a positive constant \(C_{q, \ell}\) such that

\[
|B_{L^k, h^{(k)}} - \frac{1}{(2\pi)^m} \sum_{r=0}^{q} \frac{\phi_r}{k^r}|_{C^\ell} \leq C_{q, \ell} \frac{k^{q+1}}{k^{q+1}},
\]

where \(| \cdot |_{C^\ell}\) stands for the usual \(C^\ell\)-norm.

Moreover, each constant \(C_{q, \ell}\) is uniform with respect to the \(C^s\)-norm of \(\omega\), with \(s = \ell + 2(m + q + 2)\), and (8.2.5) then holds when the \(C^\ell\)-norm in (8.2.5) includes the derivatives of \(\omega\) up to the order \(\ell\).

Finally, the first two coefficients \(\phi_0, \phi_1\) in (8.2.4) are given by

\[
\phi_0 \equiv 1, \quad \phi_1 = \frac{s_g}{4},
\]

where \(s\) denotes the scalar curvature of \((M, g, \omega)\).

**Proof.** The fact that the family of Bergman densities \(B_{L^k, h^{(k)}}\) admits an asymptotic expansion as above can be regarded as a special case of a more general theorem due to L. Boutet de Monvel and J. Sjöstrand [40]. In the current setting, a weak form of Theorem 8.2.1, namely the estimate (8.2.5) for \(q = 0\) and \(\ell = 2\), is due to G. Tian [182] — cf. also [37], [38], [65]; the above full asymptotic assumption has been first given by S. Zeldich [200] and D. Catlin [52], cf. also [169]. The explicit determination of the coefficients \(\phi_0\) and \(\phi_1\) is due to Z. Lu [138], which also includes the explicit determination of \(\phi_2\) and \(\phi_3\). The formulation of Theorem 8.2.1 given here is extracted from the more general Theorems 4.5 and 4.6 in [140]). The proof of Theorem 8.2.1 goes far beyond the scope of these notes: we refer the reader to the above references and to [30].

As a direct consequence of Theorem 8.2.1 we get:
Proposition 8.2.1. Let $M$ be any compact Kähler manifold $(M, g, J, \omega)$ of complex dimension $m$, polarized by a hermitian holomorphic line bundle $(L, h)$. Then, the dimension $d(L^k) = N_k + 1$ of $H^0(M, L^k)$ has the following asymptotic expansion

$$d(L^k) \simeq \frac{1}{(2\pi)^m} \sum_{r=0}^{+\infty} a_r k^{m-r},$$

meaning that, for any non-negative integer $q$, we have

$$d(L^k) = \frac{1}{(2\pi)^m} \sum_{r=0}^{q} a_r k^{m-r} + O(k^{m-r-1}),$$

with $a_r = \int_M \phi_r v_g$. In particular,

$$a_0 = V = \int_M v_g, \quad a_1 = \frac{1}{4} \int_M s_g v_g.$$

Proof. By the very definition of the Bergman density, we have that $d(L^k) = \int_M B_{L^k, h}(\psi) v_g = k^m \int_M B_{L^k, h}(\psi) v_g$, where $g_\psi$ denotes the metric determined by $\omega_k = k \omega$. We then conclude as for Proposition 8.9.1, by using Theorem 8.2.1. □

Remark 8.2.2. For $k$ large enough, $L^k$ is very ample and all cohomology space $H^j(M, L^k)$ are reduced to $\{0\}$ except for $H^0(M, L^k)$. Then, $d(L^k)$ is equal to the holomorphic Euler characteristic $\chi(L)$, defined by

$$\chi(L) = \sum_{j=0}^{m} (-1)^j h^j(L),$$

where $h^j(L)$ denotes the (complex) dimension of $H^j(M, L)$. In general, $\chi(L)$ is expressed by the Riemann-Roch formula which can be written as

$$\chi(L) = (ch(L) \cup Td(M, J))[M],$$

where $ch(L)$ denotes the Chern character of $L$ and $Td(M, J)$ the Todd class of the tangent bundle $(TM, J)$, viewed as a complex vector bundle of rank $m$ over $M$. For $k$ large enough, we then have

$$d(L^k) = (ch(L^k) \cup Td(M, J))[M],$$

where the rhs is actually a polynomial in $k$ of degree $m$. To make this point precise, recall that the Chern character is the additive characteristic class associated with the power series $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$, whereas the Todd class is the multiplicative characteristic class associated to the power series $\frac{e^x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{j=1}^{\infty} (-1)^{j-1} \frac{B_{2j}}{(2j)!} x^{2j}$, where the $B_{2j}$ are positive rational numbers, namely the so-called Bernoulli numbers. For any complex vector bundle $E$ of rank $r$ over $M$, we then have: $ch(E) = \sum_{j=1}^{r} e^{\gamma_j}$ and $Td(E) = \prod_{j=1}^{r} \frac{\gamma_j}{1-e^{-\gamma_j}}$, where the $\gamma_j$’s denote the Chern roots, formally defined by $c(E) = \prod_{j=1}^{r} (1 + \gamma_j)$, where $c(E) = 1 + \sum_{j=1}^{r} c_j(E)$ denotes the total Chern class of $E$; the Chern roots $\gamma_j$ defined that way don’t live in $H^2(M, \mathbb{Z})$, only in some extension of $H^2(M, \mathbb{Z})$, but each symmetric polynomial function of the $\gamma_j$’s is a well-defined polynomial function of the Chern classes $c_j(E)$, hence a well-defined element in $H^*(M)$ (for all the material mentioned in this
remark, we refer the reader to [100, Chapter 1]). In the current situation, where \( L \) is of rank 1, the (unique) Chern root \( \gamma_1 \) is equal to the first Chern class \( c_1(L) \), which is represented by \( \frac{\omega}{2\pi} \), as the Kähler structure is polarized by \( L \); similarly, the first Todd class \( Td_1(M, J) \) is equal to \( \frac{1}{2}c_1(M, J) \), which is represented by \( \frac{\omega}{2\pi} \), where \( \rho \) denotes the Ricci form of the Kähler structure (cf. Section 1.19). The rhs of (8.2.12) is then equal to \( \int_M (1 + \sum_{i=1}^m k^i (\frac{\omega}{2\pi})^i)(1 + \frac{\rho}{4\pi} + \ldots) \), where only forms of degree \( m \) contribute non trivially. The right hand side is then a polynomial of degree \( m \) in \( k \), whose leading coefficient is equal to \( \frac{1}{(2\pi)^m} \int_M \frac{\omega^m}{m!} \) and the second coefficient is equal to \( \frac{1}{(2\pi)^m} \int_M \frac{1}{2} \rho \wedge \frac{\omega^{m-1}}{(m-1)!} \). We thus retrieve (8.2.9).

Theorem 8.2.1 plays a prominent rôle in this chapter, as will be clear in the next sections. In particular, Theorem 8.2.1 appears as a crucial tool in the proof of Donaldson’s Theorem 8.7.1. A weighted version of Theorem 8.2.1 plays a similar rôle in Mabuchi’s improvement of Theorem 8.7.1 quoted in Remark 8.7.3.

### 8.3. The Kodaira embedding theorem

We first introduce the canonical map, \( \varphi_L \), associated to any pair \((M, L)\) formed of a (connected) compact complex manifold \((M, J)\) and a holomorphic line bundle \( L \) over \( M \) (no Kähler metric on \( M \) is needed at this level).

Denote by \( E = H^0(M, L) \) the (complex) vector space of holomorphic sections of \( L \); we assume that its dimension is positive, equal to \( N + 1 \). Denote \( E^* \) the (complex) dual of \( E \) and by \( \mathbb{P}(E^*) \) the corresponding (complex) projective space. We denote by \( D_L \) the set of points \( x \) of \( M \) such that \( s(x) = 0 \) for all \( s \) in \( E \), and by \( M_0 \) the open subset \( M \setminus D_L \).

A point \([\alpha]\) of \( \mathbb{P}(E^*) \) can be viewed either as a complex line in \( E^* \) or as a complex hyperplane in \( E \), namely the kernel, \( H_{[\alpha]} = (E^*/[\alpha])^* \), of any non-zero element \( \alpha \) of \( E^* \) in the complex line \([\alpha]\). Both viewpoints are useful and will be used indifferently. In particular, the tautological line bundle \( \Lambda \) over \( \mathbb{P}(E^*) \) is naturally described by:

\[
\Lambda_{[\alpha]} = [\alpha] \subset E^*,
\]

whereas the dual tautological line bundle \( \Lambda^* = \mathcal{O}(1) \) is best described by

\[
\Lambda^*_{[\alpha]} = E/H_{[\alpha]},
\]

for any \([\alpha]\) in \( \mathbb{P}(E^*) \).

The canonical map \( \varphi_L : M_0 \to \mathbb{P}(E^*) \) is the holomorphic map defined as follows:

\[
\varphi_L(x) = \{ s \in E \mid s(x) = 0 \},
\]

for any \( x \) in \( M_0 \) (since \( x \) sits in \( M_0 = M \setminus D_L \), the rhs of (8.3.3) is a complex hyperplane in \( E \), hence determines an element of \( \mathbb{P}(E^*) \) in the second acceptation).

For any \( x \) in \( M_0 \) denote par ev\(_x\) : \( E \to L_x \), the evaluation map at \( x \), defined by: \( ev_x(s) = s(x) \). The map \( \varphi_L \) is then alternatively defined by

\[
\varphi_L(x) = ev^*_x(L^*_x),
\]
where $ev^*_x : L^*_x \rightarrow E^*$ denotes the (algebraic) dual map of $ev_x$ and $ev^*_x(L_x^*)$ the image of $L_x^*$ in $E^*$ (notice that $ev_x$ is surjective for any $x$ in $M_0$, so that $ev^*_x$ is injective).

Whichever formulation we adopt to define $\varphi_L$, we get the following natural identification

\begin{equation}
\varphi_L^*(\Lambda^*) = L|_{M_0},
\end{equation}

of the restriction of $L$ to $M_0$ with the pull-back of the dual tautological line bundle $\Lambda^*$ by $\varphi_L$. This means that

\begin{equation}
L_x \cong \Lambda^*_{\varphi_L(x)} = E/H_{\varphi_L(x)}^*,
\end{equation}

for any $x$ in $M_0$; to any $u$ in $L_x$ we associate the (well-defined) element $[s]$ in $E/H_{\varphi_L(x)}^*$ such that $s(x) = u$. We similarly get a natural identification

\begin{equation}
\varphi_L^*(\Lambda) = L^*|_{M_0},
\end{equation}

i.e. a natural isomorphism

\begin{equation}
L^*_x \cong \Lambda_{\varphi_L(x)} = \varphi_L(x) \subset E^*,
\end{equation}

for each $x$ in $M_0$, defined as follows: to any $\zeta$ in $L_x^*$ we associate $ev^*_x(\zeta)$. The link between these two viewpoints is given by the duality identity:

\begin{equation}
\langle ev^*_x(\zeta), s \rangle = \langle \zeta, s(x) \rangle
\end{equation}

for any $s$ in $E$ and any $\zeta$ in $L_x^*$.

For any $x$ in $M_0$, the tangent space $T_{\varphi_L(x)} \mathbb{P}(E^*)$ is naturally identified with the space $\text{Hom}(H_{\varphi_L(x)}, E/H_{\varphi_L(x)})$ of $\mathbb{C}$-linear homomorphisms from the corresponding hyperplane $H_{\varphi_L(x)}$ to the quotient $E/H_{\varphi_L(x)}$, cf. Section 6.1. By (8.3.6), $\text{Hom}(H_{\varphi_L(x)}, E/H_{\varphi_L(x)})$ is the same as the space $\text{Hom}(H_{\varphi_L(x)}, L_x)$. The tangent map to $\varphi_L$ at $x$ is then tautologically given by

\begin{equation}
(d\varphi_L)_x(X) = \{ s \in H_{\varphi_L(x)} \mapsto (dXs)(x) \},
\end{equation}

where $(dXs)(x)$ stands for the derivative of $s$ at $x$ along $X$ (this makes sense, as $s(x) = 0$; e. g. $(dXs)(x) = (\nabla_X s)(x)$ for any $\mathbb{C}$-linear connection on $L$).

For any basis $s = \{s_0, s_1, \ldots, s_N\}$ of $E$, denote by $\sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_N\}$ the (algebraic) dual basis of $E^*$, so that $\langle \sigma_i, s_j \rangle = \delta_{ij}$, for any $i, j$ in $0, 1, \ldots, N$. For any such choice, we get the following expression of $\varphi_L$:

\begin{equation}
\varphi_L(x) = \left[ \sum_{i=0}^{N} s_j(x) \otimes \sigma_j \right] \subset L_x \otimes E^*,
\end{equation}

via the natural identification $\mathbb{P}(L_x \otimes E^*) \cong \mathbb{P}(E^*)$ for any $x$ in $M_0$. Indeed, for any $s$ in $E$ and any $x$ in $M_0$, we evidently have that

\begin{equation}
\langle \sum_{i=0}^{N} s_j(x) \otimes \sigma_j, s \rangle = s(x),
\end{equation}

so that the kernel of $\sum_{i=0}^{N} s_j(x) \otimes \sigma_j$ in $E$ coincides with $H_{\varphi_L(x)}$. Note that the expression $\sum_{i=0}^{N} s_j(x) \otimes \sigma_j$ appearing in the rhs of (8.3.11) is independent of the choice of the basis $s$ of $E$. 

On the other hand, \( s \) identifies \( \mathbb{P}(E^*) \) with the standard projective space \( \mathbb{P}^N \) via the map \( \hat{s} : \mathbb{P}(E^*) \to \mathbb{P}^N \) defined by

\[
\hat{s}(\sigma) = (\langle \sigma, s_0 \rangle : \langle \sigma, s_1 \rangle : \ldots : \langle \sigma, s_N \rangle),
\]

for any non-zero \( \sigma \) in \( E^* \), where \([\sigma]\) denotes the corresponding element of \( \mathbb{P}(E^*) \). By substituting \([\sigma]\) = \( \sum_{j=0}^{N}[s_j(x) \otimes \sigma_j] \) in \( \mathbb{P}(L_\ell \otimes E^*) = \mathbb{P}(E^*) \), we thus get

\[
j_\ell(x) := \hat{s} \circ \varphi_L(x) = (s_0(x) : s_1(x) : \ldots : s_N(x)),
\]

where the \( s_0(x), s_1(x), \ldots, s_N(x) \), which are elements of the fibre \( L_x \), are here regarded as the homogeneous coordinates of \( j_\ell(x) \) in \( \mathbb{P}^N \) via any auxiliary (holomorphic) local trivialization, \( \sigma \), of \( L \) in a neighbourhood of \( x \); if \( s_i = \lambda_i \sigma \), \( i = 0, 1, \ldots, N \), where each \( \lambda_i \) is a holomorphic function defined on this neighbourhood, we then have

\[
j_\ell(x) = (\lambda_0(x) : \lambda_1(x) : \ldots : \lambda_N(x)).
\]

\textbf{Definition 14.} Let \((M, J)\) be a compact complex manifold and \( L \) a holomorphic line bundle over \( M \). Then, \( L \) is said to be \textit{very ample} if the map \( \varphi_L \) is defined everywhere — i.e. if \( M_0 = M \) — and if \( \varphi_L \) is an embedding from \( M \) to \( \mathbb{P}((H^0(M, L))^*) \).

If \( V \subset E = H^0(M, L) \) is a (complex) non-zero vector subspace of \( H^0(M, L) \), we can similarly define a holomorphic map from \( M \setminus D_V \) to \( \mathbb{P}(V^*) \), where \( D_V \) denotes the set of points \( x \) of \( M \) such that \( s(x) = 0 \) for all \( s \) in \( V \). We then have

\textbf{Proposition 8.3.1.} A holomorphic line bundle \( L \) over a compact complex manifold is very ample if and only if there exists a complex vector subspace \( V \) of \( E = H^0(M, L) \) such that \( \varphi_V \) is defined everywhere and is an embedding from \( M \) to \( \mathbb{P}(V^*) \).

\textbf{Proof.} Let \( V \) be any complex vector subspace of \( E = H^0(M, L) \) such that \( \varphi_V \) is defined everywhere — i.e. \( D_V = \emptyset \) — and is an embedding. Since \( D_V = \emptyset \), \( D_L = \emptyset \) for a fortiori, so that \( \varphi_L \) is defined everywhere as well. Moreover, the image of \( \varphi_L \) misses the projective subspace \( \mathbb{P}((E/V)^*) \), where \( (E/V)^* \) is viewed as the subspace of elements \( \alpha \) of \( E^* \) such that \( \alpha|_V \equiv 0 \). Now, we have a natural submersion, \( p_V \), from \( \mathbb{P}(E^*) \setminus \mathbb{P}((E/V)^*) \) onto \( \mathbb{P}(V^*) \) and we then have \( \varphi_V = p_V \circ \varphi_L \), i.e. the following commutative diagram:

\[
M \xrightarrow{\varphi_L} \mathbb{P}(E^*) \setminus \mathbb{P}((E/V)^*) \xleftarrow{p_V} \mathbb{P}(V^*)
\]

It is then clear that \( \varphi_L \) is an embedding from \( M \) to \( \mathbb{P}(E^*) \setminus \mathbb{P}((E/V)^*) \), hence to \( \mathbb{P}(E^*) \), whenever \( \varphi_V \) is an embedding from \( M \) to \( \mathbb{P}(V^*) \). \( \square \)

\textbf{Remark 8.3.1.} If \( L \) is a very ample holomorphic line bundle over a compact complex manifold of (complex) dimension \( m \), it can be shown that there always exists a subspace \( V \) of \( H^0(M, L) \) of (complex) dimension smaller than or equal to \( 2(m + 1) \) such that \( \varphi_V \) is an embedding from \( M \) to \( \mathbb{P}(V^*) \), cf. [66] Proposition 11.9 for a stronger statement.
Example 1. (i) By Proposition 6.1.1, for any complex projective space \( \mathbb{P}(V) \), \( H^0(\mathbb{P}(V), \mathcal{O}(1)) = V^* \), so that \( (H^0(\mathbb{P}(V), \mathcal{O}(1)))^* = V \) and \( \varphi_{\mathcal{O}(1)} \) is then the identity map from \( \mathbb{P}(V) \) to itself. More generally, for any \( k > 0 \), \( H^0(\mathbb{P}(V), \mathcal{O}(k)) = \bigotimes^k(V^*) \). \( (H^0(\mathbb{P}(V), \mathcal{O}(k)))^* = \bigotimes^k V \) and \( \varphi_{\mathcal{O}(k)} \) is then the Veronese embedding from \( \mathbb{P}(V) \) to \( \mathbb{P}(\bigotimes^k V) \) defined by

\[
(8.3.17) \quad x \mapsto x^{\otimes k},
\]
for any complex line \( x \) in \( E \). In particular, the Veronese embedding associated to \( \mathcal{O}(k) \) over the complex projective line \( \mathbb{P}^1 \) is the holomorphic embedding from \( \mathbb{P}^1 \) to \( \mathbb{P}^k \) defined, in term of homogeneous coordinates, by

\[
(8.3.18) \quad (z : t) \mapsto (z^k : z^{k-1}t : \ldots : zt^{k-1} : t^k).
\]

(ii) Let \( M \) be any (compact) projective manifold and let \( \varphi : M \to \mathbb{P}(V) \) be a holomorphic full embedding from \( M \) to some complex projective space \( \mathbb{P}(V) \) (here, full means that \( \varphi(M) \) is not contained in a proper projective subspace of \( \mathbb{P}(V) \)). Denote by \( L = \varphi^* \mathcal{O}(1) \) the restriction of \( \mathcal{O}(1) \) to \( M \). Then, \( L \) is very ample. This is a consequence of Proposition 8.3.1, by choosing \( V = H^0(\mathbb{P}(V), \mathcal{O}(1)) \subset E = H^0(M, L) \); then, \( \varphi_V = \varphi \) which is an embedding by assumption (notice that \( H^0(\mathbb{P}(V), \mathcal{O}(1)) \subset H^0(M, L) \) is a proper inclusion in general, cf. [93] for a more precise discussion on this point).

The \textit{Kodaira embedding theorem} can be formulated as follows:

**Theorem 8.3.1** (K. Kodaira [122]). Let \( (M, J) \) be a compact complex manifold and let \( L \) a holomorphic line bundle over \( M \). Then \( L \) is ample if and only if there exists a positive integer \( k_0 \) such that \( L^k \) is very ample for any integer \( k \geq k_0 \).

**Proof.** In one direction, the theorem is clear: if \( L^k \) is very ample, for some positive integer \( k \), then \( (M, J) \) is holomorphically embedded in the complex projective space \( \mathbb{P}(H^0(M, L^k)^*) \) via the map \( \varphi_{L^k} \) and \( L^k \) is then identified with the restriction of the dual tautological line bundle of \( \mathbb{P}(H^0(M, L^k)^*) \); since the latter is ample, \( L^k \) is then ample, as well as \( L \) itself (more details on the induced Fubini-Study Kähler metrics below). The difficult part of the theorem is to show that if \( L \) is ample, for \( k \) large enough the map \( \varphi_{L^k} \) is defined everywhere and is an embedding, i.e. an injective immersion as \( M \) is compact. By the very definition (8.3.3)-(8.3.4) of the Kodaira map and the expression (8.3.10) of its derivative, this amounts to checking the following two conditions:

1. For any two (distinct) points \( x, y \) of \( M \), there exits \( s \) in \( E_k \) such that \( s(x) = 0 \) and \( s(y) \neq 0 \).
2. For any \( x \) in \( M \) and any \( X \) in \( T_x M \), there exists \( s \) in \( E_k \) such that \( s(x) = 0 \) and \( (d_X s)(x) \neq 0 \).

The proof of these two facts makes use of the vanishing theorem 1.22.5 and of the sheaf theory machinery, which goes outside the scope of these notes. For a complete proof, we refer the reader to the original paper [122] or to more recent references like [53], [93], [193], [66], [140]. □
Remark 8.3.2. As observed in [200] the fact that the canonical map \( \varphi_L \) is defined everywhere when \( L \) is ample and \( k \) is large enough directly follows from Theorem 8.2.1. Indeed, when \( k \) is large, it follows from this theorem that the Bergman density \( B_{L,k,h(k)} \) is everywhere positive, meaning that the set \( D_L \) is empty, cf. also [140].

Let \( L \) be a very ample holomorphic line bundle over a compact (projective) complex manifold \( M \) and, as before, denote by \( \varphi_L \) the canonical map from \( M \) to the complex projective space \( \mathbb{P}(E^*) \) — with \( E = H^0(M,L) \) — which, by Theorem 8.3.1, is a holomorphic embedding. We then have:

**Proposition 8.3.2.** The group \( \text{Aut}_L(M,J) \) coincides with the group of those elements of \( \text{Aut}(\mathbb{P}(E^*)) \) which preserve the image of \( M \) by \( \varphi_L \).

**Proof.** Let \( \Phi \) be any element of \( \text{Aut}_L(M,J) \). Then, up to the overall action of \( \mathbb{C}^* \), \( \Phi \) determines a linear automorphism, \( \hat{\Phi} \), of \( E \), defined by \( s \mapsto (\hat{\Phi}(s))(x) := \hat{\Phi}(s(\Phi^{-1} \cdot x)) \), which, in turn, determines a well-defined automorphism, \( \hat{\Phi} \) say, of \( \mathbb{P}(E^*) \). For any \( x \) in \( M \), we then have: \( \Phi(\varphi_L(x)) = \hat{\Phi}(\Phi^*(\varphi_L(x))) = \hat{\Phi}(\Phi^*(\varphi_L(x))) = \varphi_L(\Phi \cdot x) \). This means that \( \hat{\Phi} \) preserves \( M \) and that the restriction of \( \hat{\Phi} \) to \( M \) coincides with \( \Phi \). Conversely, any automorphism \( \Psi \) of \( \mathbb{P}(E^*) \) is determined by a linear automorphism of \( E^* \) — cf. the proof of Proposition 6.1.2 — hence determines an automorphism \( \hat{\Psi} \) of the dual tautological line bundle \( \Lambda^* \) of \( \mathbb{P}(E^*) \), defined up to the action of \( \mathbb{C}^* \); if \( \hat{\Psi} \) preserves \( M \) and induces \( \Phi \) in \( \text{Aut}(M,J) \), then \( \hat{\Psi} \) restricts to an automorphism of \( L \) over \( \Phi \), defined up the action of \( \mathbb{C}^* \), so that \( \Phi \) is an element of \( \text{Aut}_L(M,L) \), cf. (8.3.5). The two constructions are clearly inverse to each other. \( \square \)

Remark 8.3.3. In the context of the Kodaira map \( \varphi_L \), the Bergman density defined by (8.2.2) can be alternatively defined as follows, cf. [81]:

\[
B_{L,h}(x) = \max_{s \in H^0(M,L)} \frac{|s(x)|^2}{\int_M |s|^2 h} \quad \text{for any } x \in M.
\]

Indeed, for any orthonormal basis \( s = \{s_0, s_1, \ldots, s_N\} \) of \( H^0(M,L) \), any \( s \) in \( H^0(M,L) \) such that \( \int_M |s|^2 h = 1 \) is of the form \( s = \sum_{i=0}^N a_i s_i \), with \( \sum_{i=0}^N |a_i|^2 = 1 \). Then, \( |s(x)|^2 \leq \left( \sum_{i=0}^N a_i^2 \right) \sum_{i=0}^N |s_i(x)|^2 = B_{L,h}(x) \) (by using the hermitian Cauchy–Schwarz inequality). Conversely, let \( \{s_0, s_1, \ldots, s_N\} \) be any orthonormal basis of \( H^0(M,L) \) such that \( s_1(x) = \cdots = s_N(x) = 0 \) (if \( x \in D_L \) this holds for any basis of \( H^0(M,L) \); if \( x \notin D_L \), choose any orthonormal basis \( \{s_0, \ldots, s_N\} \) of the hyperplane \( \varphi_L(x) \) of \( H^0(M,L) \) and complete it into an orthonormal basis of \( H^0(M,L) \) by adding \( s_0 \)); then, \( B_{L,h}(x) = |s_0(x)|^2 \).

**8.4. Bergman Metrics on Polarized Manifolds**

The canonical embedding \( \varphi_L \) attached to any very ample holomorphic line bundle \( L \) over \( M \) is defined without using any metric on \( M \) or any hermitian inner product on \( L \) or any basis of \( E = H^0(M,L) \). On the other hand, any choice of a basis \( s = \{s_0, \ldots, s_N\} \) of \( E \) determines a (positive definite) hermitian inner product, \( \langle \cdot, \cdot \rangle_s \), on \( E \), namely the unique one for which \( s \) is orthonormal. This, in turn, induces a hermitian inner product...
on \( E^* \), also denoted by \( \langle \cdot, \cdot \rangle_h \) for simplicity, hence a Fubini-Study metric on \( \mathbb{P}(E^*) \) — of holomorphic sectional curvature equal to 2, cf. Remark 8.1.1 — polarized by the dual tautological line bundle \( \Lambda^* \) of \( \mathbb{P}(E^*) \), equipped by the (fiberwise) hermitian inner product induced by the hermitian inner product \( \langle \cdot, \cdot \rangle_{sL^*} \) of \( E^* \). The pull-back of this metric by the Kodaira embedding \( \varphi_L \) is then a Kähler metric, denoted by \( g_{M,s} \), whose Kähler form, \( \omega_{M,s} \), is the curvature form of the (pointwise) hermitian inner product, \( h_{sL} \) of \( L \) induced by the above hermitian inner product of \( \Lambda^* \) via the isomorphism \( L = \varphi_L^*(\Lambda^*) \).

We then have:

**Lemma 8.4.1.** For any basis \( s = \{s_0, s_1, \ldots, s_N\} \) of \( E \), the induced hermitian inner product \( h_{sL} \) on \( L \) at any point \( x \) of \( M \) is given by

\[
(8.4.1) \quad |u|_{h_{sL}}^2 = \frac{|u|_h^2}{\sum_{i=0}^N |s_i(x)|_{h_s}^2},
\]

for any \( u \) in \( L_x \) and any auxiliary hermitian inner product \( h \) on \( L \). Equivalently, \( h_{sL} \) is the unique hermitian inner product on \( L \) such that

\[
(8.4.2) \quad \sum_{i=0}^N |s_i(x)|_{h_{sL}}^2 = 1, \quad \text{for any } x \text{ in } M.
\]

**Proof.** We first consider the induced hermitian inner product on \( L_x^* \), for each \( x \) in \( M \), via the identification of \( L^* \) with \( \varphi_L^* \Lambda \) given by (8.3.8). Any \( \zeta \) in \( L_x^* \) is then identified with \( \text{ev}_x^*(\zeta) \) in \( E^* \), whose explicit expression is given by (8.3.9). From (8.3.9) we readily infer

\[
(8.4.3) \quad |\zeta|_{h_{sL}}^2 := |\text{ev}_x^*(\zeta)|_{L_x^*}^2 = \sum_{i=0}^N |\langle \text{ev}_x^*(\zeta), s_i \rangle|^2 = \sum_{i=0}^N |\langle \zeta, s_i(x) \rangle|^2,
\]

for any \( x \) in \( M \) and any \( \zeta \) in \( L_x^* \). The induced hermitian inner product on \( L \) is then given by

\[
(8.4.4) \quad |u|_{h_{sL}}^2 = \frac{|\langle \zeta, u \rangle|^2}{|\zeta|_{h_{sL}}^2} = \frac{|\langle \zeta, u \rangle|^2}{\sum_{i=0}^N |\langle \zeta, s_i(x) \rangle|^2},
\]

for any non-zero \( \zeta \) in \( L_x^* \). In particular, for any \( j = 0, 1, \ldots, N \), we have that

\[
(8.4.5) \quad |s_j(x)|_{h_{sL}}^2 = \frac{|\langle \zeta, s_j(x) \rangle|^2}{\sum_{i=0}^N |\langle \zeta, s_i(x) \rangle|^2}.
\]

We then get (8.4.2), hence (8.4.1).

Lemma 8.4.1 can be reformulated as follows:

**Proposition 8.4.1.** Let \( (M, L) \) be a compact Kähler manifold polarized by a (ample) holomorphic line bundle \( L \). Suppose that \( L \) is very ample and denote by \( \varphi_L \) the Kodaira embedding of \( M \) in \( \mathbb{P}(E^*) \), with \( E = H^0(M, L) \). For any basis \( s = \{s_0, s_1, \ldots, s_N\} \) of \( E \), consider the Fubini-Study metric on \( \mathbb{P}(E^*) \) determined by the hermitian inner product \( \langle \cdot, \cdot \rangle_{sL^*} \) of \( E^* \), as defined above, and denote by \( \omega_{M,s} \) the Kähler form of the induced Kähler metric \( g_{M,s} \) on \( M \) via \( \varphi_L \). For any hermitian inner product \( h \) on \( L \), denote by \( \omega_h \)
the curvature form of the Chern connection of \((L,h)\). Then, \(\omega_{M,s}\) and \(\omega_h\) are related by

\[
\omega_{M,s} = \omega_h + \frac{1}{2} \ddc \log \left( \sum_{i=0}^{N} |s_i(x)|_h^2 \right).
\]

**Proof.** If \(\tilde{\pi}\) denotes the natural projection of \(\tilde{L}\) to \(M\) as in (8.1.3), we have \(\tilde{\pi}^*(\omega_h) = -\frac{1}{2} \ddc \log r_h^2\), where \(r_h^2\) denotes the square norm function on \(L\) determined by \(h\), whereas \(\tilde{\pi}^*\omega_{M,s} = -\frac{1}{2} \ddc \log r_{h_s}^2\), where \(r_{h_s}^2\) denotes the square norm function on \(L\) determined by \(h(s)\). We then readily infer (8.4.6) from (8.4.1).

**Remark 8.4.1.** In (8.4.6), the Kähler form \(\omega_{M,s}\) only depends on the choice of the basis \(s\) of \(E\) and it is easy to check directly that the right hand side of (8.4.6) is actually independent of \(h\). Also observe that \(\omega_h\) need not be positive — although it can be chosen so by an appropriate choice of \(h\), as \(L\) is ample — and that the very ampleness of \(L\) implies that \(\sum_{i=0}^{N} |s_i(x)|_h^2\) has no zero on \(M\). Observe finally that \(\sum_{i=0}^{N} |s_i(x)|_h^2\) is not, in general, the Bergman density of \(\omega_{M,s}\), nor of \(\omega_h\), when the latter is positive.

**Theorem 8.4.1** (G. Tian [182]). Let \((M,g,J,\omega)\) be a compact Kähler manifold of complex dimension \(m\), polarized by a hermitian holomorphic line bundle \((L,h)\), and suppose that \(L^k\) is very ample for all \(k \geq k_0\). For any \(k \geq k_0\), consider: the hermitian inner product on \(E_k = H^0(M,L^k)\) determined by the induced pointwise hermitian product \(h^{(k)}\) and the volume form \(v_{\omega} = k^m \nu_{g}\) corresponding to the polarization of \(M\) by \((L^k,h^{(k)})\), as in (8.2.1); the induced Fubini-Study metric of \(\mathbb{P}(E_k)\); the Kähler form \(\omega_{M,s^{(k)}}\) of the Kähler metric induced by the canonical embedding \(\varphi_{L^k}\), where \(s^{(k)}\) stands for any orthonormal basis of \(E_k\) with respect to the above hermitian inner product; the renormalized induced Kähler form \(\tilde{\omega}_{FS,k} = \frac{1}{k} \omega_{M,s^{(k)}}\). Then, when \(k\) tends to \(+\infty\) the sequence \(\tilde{\omega}_{FS,k}\) \(C^\infty\) converges to \(\omega\).

**Proof.** From Proposition 8.4.1 applied to the pair \((M,L^k)\), we get

\[
\tilde{\omega}_{FS,k} = \omega + \frac{1}{2k} \ddc \log B_{L^k,h^{(k)}},
\]

for any \(k \geq k_0\), where \(B_{L^k,h^{(k)}}\) denotes the Bergman density of \((M,(L^k,h^{(k)}))\), as defined in Section 8.2. By Theorem 8.2.1, \(B_{L^k,h^{(k)}}\) is \(C^\infty\) bounded from above when \(k\) tends to \(+\infty\): \(\frac{1}{2k} \ddc \log B_{L^k,h^{(k)}}\) then \(C^\infty\) converges to zero.

**Remark 8.4.2.** As already mentioned in the proof of Theorem 8.2.1, Theorem 8.4.1 has been first obtained by G. Tian as a consequence of a weak form of Theorem 8.2.1 established in [182], also cf. [37], [38].

**Remark 8.4.3.** The metrics \(\tilde{\omega}_{FS,k} = \frac{1}{k} \omega_{M,s^{(k)}}\) appearing in Theorem 8.4.1 all belong to the Kähler class \(\Omega = 2\pi c_1(L)\) for any \(k \geq k_0\). These metrics are called Bergman metrics or algebraic Kähler metrics, cf. [80], relative to the ample line bundle \(L\). Tian’s theorem can then be roughly reformulated as follows: The space of Bergman metrics relative to \(L\) is dense in \(\mathcal{M}_{2\pi c_1(L)}\).
For any basis \( s \) of \( E = H^0(M, L) \), the induced (fiberwise) hermitian inner product \( h_s \) on \( L \) determines in turn a hermitian inner product on \( E \) defined by

\[
\langle s, \tilde{s} \rangle_{h_s} = \int_M \langle s(x), \tilde{s}(x) \rangle_{h_s} v_{M,s},
\]

for any two elements \( s, \tilde{s} \) of \( E \), where \( v_{M,s} = \frac{1}{\text{vol}(M)} \omega_{M,s} \) denotes the volume form of \( \omega_{M,s} \), cf. (8.2.1).

In the case when \((M, L) = (\mathbb{P}(V), \Lambda^*)\) we have \( E = V^* \) — cf. Proposition 6.1.1 — and \( \varphi_L \) is then the identity map from \( M = \mathbb{P}(V) \) to \( \mathbb{P}(E^*) \). Then, the global scalar product \( \langle \cdot, \cdot \rangle_{h_s} \) defined by (8.4.8) coincides, up to normalization, with the previously defined hermitian inner product \( \langle \cdot, \cdot \rangle_s \) of \( H^0(\mathbb{P}(V), \Lambda^*) \), with respect to which \( s \) is orthonormal, cf. below in this section. This, however, is no longer true in general and we now make this point more precise.

Denote by \( \mathcal{B}(E) \) the space of bases of \( E = H^0(M, L) \), by \( \mathcal{B}_0(E) = \mathcal{B}(E)/\mathbb{C}^* \) the set of bases of \( E \) modulo homothety and by \( \pi \) the natural projection from \( \mathcal{B}(E) \) to \( \mathcal{B}_0(E) \). Both spaces \( \mathcal{B}(E) \) and \( \mathcal{B}_0(E) \) have a natural complex structure and \( \pi \) makes \( \mathcal{B}(E) \) into a holomorphic \( \mathbb{C}^* \)-principal bundle over \( \mathcal{B}_0(E) \).

The complex manifold \( \mathcal{B}(E) \) admits a simply transitive right action of the complex linear group \( GL(N + 1, \mathbb{C}) \), defined by

\[
(s \cdot \gamma)_i = \sum_{j=0}^N \gamma_{ji} s_j, \quad i = 0, \ldots, N,
\]

for any \( s \) in \( \mathcal{B}(E) \) and any \( \gamma \) in \( GL(N + 1, \mathbb{C}) \). This action clearly descends to a (right) action of \( GL(N + 1, \mathbb{C}) \), in fact of \( PGL(M + 1, \mathbb{C}) = GL(N + 1, \mathbb{C})/\mathbb{C}^* \), on \( \mathcal{B}_0(E) \), whereas \( \mathcal{B}_0(E) \) admits in addition a left action of the group \( H_L(M, J) = H_{\text{red}}(M, J) \), cf. Section 8.1, determined by

\[
(\Phi \cdot s)_i(x) = \tilde{\Phi}(s_i(\Phi^{-1}(x))), \quad i = 0, \ldots, N,
\]

for any \( s \) in \( \mathcal{B}(E) \), any \( x \) in \( M \) and any \( \Phi \) in \( H_L(M, J) \), where \( \tilde{\Phi} \) denotes any lift of \( \Phi \) in \( \text{Aut}_0(L) \). Since \( \tilde{\Phi} \) is well-defined up to a constant in \( \mathbb{C}^* \), (8.4.10) determines a well-defined (left) action on \( \mathcal{B}_0(E) \), which clearly commutes with the (right) action of \( PGL(N + 1, \mathbb{C}) \).

For any \( s \) in \( \mathcal{B}(E) \) we denote by \( j_s \) the embedding of \( M \) into the standard projective space \( \mathbb{P}^N = \mathbb{P}(\mathbb{C}^{N+1}) \), defined by

\[
j_s = \tilde{s} \circ \varphi_L,
\]

where \( \tilde{s} : \mathbb{P}(E^*) \to \mathbb{P}^N \) is defined by (8.3.13); equivalently:

\[
\tilde{s}(x) = (s_0(x) : \ldots : s_N(x)),
\]

for any \( x \) in \( M \), where \( (s_0(x) : \ldots : s_N(x)) \) has to be thought of as the element \( (\frac{s_0(x)}{\sigma(x)} : \ldots : \frac{s_N(x)}{\sigma(x)}) \) of \( \mathbb{P}(\mathbb{C}^{N+1}) \) for any non-vanishing holomorphic local section of \( L \) in some neighbourhood of \( x \), cf. (8.3.14)–(8.3.15). Notice that \( j_s \) only depends on the projection of \( s \) in \( \mathcal{B}_0(E) \). We then have

\[
j_s \gamma = \gamma \circ j_s,
\]
for any $\gamma$ in $GL(N+1,\mathbb{C})$ — where $\gamma^t$ denotes the transpose of $\gamma$, defined by: $\gamma^t_{ij} = \gamma_{ji}$ — which intertwines the right action of $GL(N+1,\mathbb{C})$ on $\mathcal{B}(E)$ and the standard left action of $GL(N+1,\mathbb{C})$ on $\mathbb{P}^N$, and

$$j_{\Phi,s} = j_s \circ \Phi^{-1},$$

(8.4.14)

for any $\Phi$ in $H_L(M,J)$. In terms of the embedding $j_s : M \to \mathbb{P}^N$, the Kähler form $\omega_{M,s}$ appearing in Proposition 8.4.1 and the corresponding volume form $v_{M,s}$ are given by

$$\omega_{M,s} = j_s^* \omega_{FS}, \quad v_{M,s} = j_s^* v_{FS},$$

(8.4.15)

for any $s$ in $\mathcal{B}(E)$.

From (8.3.11), (6.3.16) and Lemma 8.4.1, we infer

$$\left(\mu_{FS}(j_s(x))\right)_{i,j} = i(\langle s_i(x), s_j(x)\rangle_{h_s} - \frac{\delta_{i,j}}{N+1}),$$

(8.4.16)

for any $x$ in $M$, where $\mu_{FS}$ denotes the momentum map $\mu_{FS} : \mathbb{P}^N \to su(N+1)$ determined by the standard action of $SU(N+1)$ on $\mathbb{P}^N$ and the Fubini-Study metric (of holomorphic sectional curvature $c = 2$) of $\mathbb{P}^N$, cf. Section 6.3. By integrating over $M$ with respect to the induced volume form $v_{M,s}$, we get:

$$\left(\int_M (\mu_{FS} \circ j_s) v_{M,s}\right)_{i,j} = i\left(\langle s_i, s_j\rangle_{h_s} - \frac{V}{N+1} \delta_{i,j}\right),$$

(8.4.17)

where $V = \int_M v_{M,s}$ denotes the total volume of $M$ for the induced metric — the same for all bases $s$ of $H^0(M,L)$, as $\omega_{M,s}$ sits in the same de Rham class as $\omega$ — and $\langle s_i, s_j\rangle_{h_s}$ is defined by (8.4.8).

In the case when $(M,L) = (\mathbb{P}(V), \Lambda^*)$, so that $\varphi_L$ is the identity, cf. above, we have that $\int_{\mathbb{P}^N} \mu_{FS} v_{FS} = 0$ — cf. (6.3.12) — and, from (8.4.17), we then infer that $\langle s_i, s_j\rangle_{h_s} = \frac{V}{N+1} \delta_{i,j}$, i.e. that $\langle \cdot, \cdot\rangle_{FS}$ coincides with the initial hermitian inner product of $E$ up to normalization. As already announced, this is no longer true in general however: by (8.4.17), the discrepancy between the hermitian inner product on $H^0(M,L)$ determined by $s$, by decreeing that $s$ is orthonormal, and the hermitian inner product defined by (8.4.8) is measured by the quantity

$$\mu_L(s) := \frac{1}{V} \int_M (\mu_{FS} \circ j_s) v_{M,s},$$

(8.4.18)

which can be viewed as the center of mass of $j_s(M) \subset \mathbb{P}^N$, when the latter is viewed as an adjoint orbit in $su(N+1)$.

Notice that $\mu_L(s)$ remains unchanged when $s$ is replaced by a homothetic basis, so that $\mu_L$ descends to a map, still denoted by $\mu_L$, from $\mathcal{B}_0(E)$ to $su(N+1)$. We then have:

Lemma 8.4.2. The map $\mu_L : \mathcal{B}_0(E) \to su(N+1)$ is equivariant under the (right) action of $SU(N+1)$, and invariant under the (left) action of $H_L(M,J)$.

Proof. For any $s$ in $\mathcal{B}(E)$ and any $\gamma$ in $SU(N+1)$, $s$ and $s\gamma$ determine the same hermitian inner product on $E$, hence Fubini-Study metric on $\mathbb{P}(E^*)$, so that $v_{M,s\gamma} = v_{M,s}$, whereas $\mu_{FS} \circ \tilde{\gamma} = Ad_{\gamma^{-1}}(\mu_{FS} \circ \tilde{s})$: we
then have $\mu_L(s\gamma) = \text{Ad}_{h^{-1}} \mu_L(s)$. This proves the first assertion. The second assertion follows from (8.4.14) and (8.4.15), from which we infer:

$$\mu_L(\Phi \cdot s) = \int_M (\Phi^{-1})^*(\mu_{FS} \circ j_s v_{M,a}) = \int_M \mu_{FS} \circ j_s v_{M,s} = \mu_L(s).$$

\[ \square \]

8.5. The Ricci form of a complex hypersurface

Let $(M, g, J, \omega)$ be a compact Kähler manifold of (complex) dimension $m \geq 2$ and let $L$ be any holomorphic line bundle over $M$. Assume that $L$ admits a holomorphic section $s$, such that the zero divisor is a smooth hypersurface $\Sigma$ of $M$. We assume that $\Sigma$ is endowed with the Kähler structure induced by the Kähler structure of $M$. In what follows, the (real) tangent bundles $TM$ and $T\Sigma$ will be considered as holomorphic vector bundles, of (complex) rank $m$ and $m - 1$ respectively, equipped with the hermitian inner product induced by $g$, cf. Section 1.8. We then have the following exact sequence of holomorphic vector bundles over $\Sigma$:

$$0 \to T\Sigma \to TM|\Sigma \to L|\Sigma \to 0,$$

where, for any $x$ in $\Sigma$, the map from $T_x M$ to $L_x$ is $(\nabla s)(x)$, the covariant derivative at $x$ of $s$ with respect to any $\mathbb{C}$-linear connection $\nabla$ on $L$ (since $s(x) = 0$, $(\nabla s)(x)$ is actually independent of the chosen connection). For any $\xi$ in $L_x$, we then denote by $X_\xi$ the unique element of $T_x M$ such that $(\nabla X_\xi s)(x) = \xi$ and which is $g$-orthogonal to $T_x \Sigma \subset T_x M$. We thus get an isomorphism of complex vector bundles over $\Sigma$:

$$TM|\Sigma = T\Sigma \oplus L|\Sigma,$$

where, at each point $x$ of $M$, $L_x$ is identified with the 1-dimensional complex subspace of $TM$ generated by $X_\xi$. This direct sum is (hermitian) orthogonal, by the very definition of $X_\xi$, and is (hermitian) isometric whenever $L|\Sigma$ is equipped with the hermitian inner product defined by

$$|\xi|^2 = |X_\xi|^2,$$

for any $x$ in $\Sigma$ and any $\sigma$ in $L_x$. We denote by $\nabla^h$ the corresponding Chern connection and by $\rho$ the curvature form of $\nabla^h$. Denote by $K_M^{-1} = \Lambda \omega TM$, resp. $K_S^{-1} = \Lambda T\Sigma$, the anti-canonical line bundle of $M$, resp. of $\Sigma$. From the exact sequence (8.5.1), we get the following isomorphism of complex line bundles over $\Sigma$:

$$(K_M^{-1})|\Sigma = K_S^{-1} \otimes L|\Sigma,$$

which is actually an isomorphism of holomorphic line bundles (in contrast with (8.5.2)). It is also an isomorphism of hermitian line bundles, provided that $L|\Sigma$ is equipped with the hermitian inner product $h$ defined by (8.5.3). Since the Ricci form of any Kähler manifold coincides with the curvature form of the anti-canonical line bundle equipped with its natural holomorphic structure and the hermitian inner product induced by the metric, we infer that the Ricci form, $\rho$, of $\Sigma$ and the Ricci form, $\rho_M$, of $M$ are related by

$$\rho = (\rho_M)|\Sigma - \rho_h.$$

In order to obtain a more workable expression, we fix an auxiliary hermitian inner product, say $h_0$, on $L$, and, for any point $x$ of $\Sigma$, we choose any local holomorphic trivialization, say $\sigma$, of $L$ on some open neighbourhood $U$ of
x, so that \( s = \lambda \sigma \) on \( U \). If \( \nabla \) denotes any \( \mathbb{C} \)-linear connection on \( L \), we thus get: \( \nabla_X s = d\lambda(X) \sigma(x) \) for any \( X \) in \( T_x M \) (since \( \lambda(x) = 0 \)). We thus have: \( d\lambda(X_\xi)s(x) = \xi \), hence \( |\xi|^2_{h_0} = |d\lambda|^2_{\text{FS}}|X_\xi|^2 |\sigma(x)|^2_{h_0} \). Since \( \xi^2_{h_0} := |X_\xi|^2_{g} \), it follows that \( h = \frac{h_0}{|\sigma|_{h_0} d\lambda g} = \frac{|\nabla s|_{h_0, g}^2}{|\sigma|_{h_0, g}} \) on \( \Sigma \). We eventually infer

\[
\rho_{\Sigma} = (\rho_M)_\Sigma - (\rho_{h_0})_\Sigma + \frac{1}{2} dd^c \log (|\nabla s|_{h_0, g}^2)_{|\Sigma}.
\]

**Example 2.** As a main example, we consider the case when \( M \) is the standard complex projective space \( \mathbb{CP}^m \), equipped with the standard Fubini-Study Kähler structure \((g_{\text{FS}}, \omega_{\text{FS}})\) — of holomorphic sectional curvature 2, cf. Remark 8.1.1 — whose Ricci form is \( \rho_{\text{FS}} = (m + 1)\omega_{\text{FS}} \), cf. (6.2.6), \( L \) is the holomorphic line bundle \( \mathcal{O}(k) \), for some positive integer \( k \), equipped with the standard hermitian inner product \( h_0 \), whose curvature form is \( \rho_{h_0} = k \omega_{\text{FS}} \), and the chosen holomorphic section \( s \) of \( L \) is therefore identified with a homogeneous polynomial of degree \( k \) defined on \( \mathbb{C}^{m+1} \), denoted by \( F \), cf. Proposition 6.1.1. We choose \( s = F \) as in Example 2, so that \( \Sigma \) is defined in \( \mathbb{CP}^{m+1} \) by the equation \( F(z_0, z_1, \ldots, z_{m+1}) = 0 \) in homogeneous coordinates. The covariant derivative of \( F \), as a section of \( \mathcal{O}(k) \), restricted to \( \Sigma \), is represented on the pull-back of \( \Sigma \) in \( \mathbb{C}^{m+2} \setminus \{0\} \) by \( dF_\Theta \). In view of (??) and of (8.5.6), we get the Tian formula:

\[
\rho_{\Sigma} = (m + 1 - k) (\omega_{\text{FS}})|_\Sigma + \frac{1}{2} dd^c \log \left( \sum_{i=0}^{m} \frac{|\partial F|^2}{|z_i|^{2k+1}} \right).
\]

### 8.6. The Donaldson gauge functional as a Kähler potential

Lemma 8.4.2 suggests that the map \( \mu_L : \mathcal{B}_0(E) \rightarrow \mathfrak{su}(N+1) \) might be a momentum map for the natural action of \( SU(N+1) \) on \( \mathcal{B}_0(E) \) — rather, on the quotient, \( \hat{\mathcal{B}}_0(E) = H_L(M) \backslash \mathcal{B}_0(E) \) — for some symplectic structure defined on \( \hat{\mathcal{B}}_0(E) \) and we now show that this is indeed the case. More precisely, we construct a Kähler structure on \( \hat{\mathcal{B}}_0(E) \), compatible with the natural complex structure, \( J_{\mathcal{B}(E)} \), of \( \mathcal{B}(E) \), in such a way that \( \mu_L \) is a momentum map for the action of \( SU(N+1) \) with respect to the corresponding Kähler form. Our treatment closely follows O. Biquard’s one in [30, Lemma 2.3].

Fix \( h_0 \) in \( \mathcal{H}^{>0}(L) \), the space of hermitian inner products on \( L \) whose Chern curvature form is positive, and denote by \( \omega_0 \) the corresponding curvature form. By definition of \( \mathcal{H}^{>0}(L) \), \( \omega_0 \) is the Kähler form of a Kähler metric on \( \mathcal{M}_Q \), with \( \Omega = 2\pi c_1(L) \). Recall — cf. Remark 8.1.2 — that the 1-form \( \tau \) on \( \mathcal{M}_Q \) defined by (4.1.2) in Section 4.1 and its primitive, the Donaldson gauge functional \( \mathbb{L}_{\omega_0} \), can be viewed as defined on \( \mathcal{H}^{>0}(L) \). Denote by \( \hat{h} \) the map from \( \mathcal{B}(E) \) to \( \mathcal{H}^{>0}(L) \) given by

\[
\hat{h} : s \mapsto h_s = \frac{h_0}{\sum_{j=0}^{N} |s_j|_{h_0}^2},
\]
Lemma 8.4.1. We thus get the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{B}(E) & \xrightarrow{\hat{h}} & \mathcal{H}^{>0}(L) \\
\downarrow \pi & & \downarrow \\
\mathcal{B}_0(E) & \xrightarrow{\mathcal{H}_{>0}(L)/\mathbb{R}} & \mathcal{M}_\omega \quad .
\end{array}
\]

In this diagram: the first vertical arrow is the \(\mathbb{C}^*\)-principal bundle map \(\pi\) from \(\mathcal{B}(E)\) to \(\mathcal{B}_0(E)\); the second vertical arrow is the \(\mathbb{R}^*\)-principal bundle map from \(\mathcal{H}_{>0}(L)\) to its quotient by the \(\mathbb{R}\)-action described in Remark 8.1.2; the third vertical arrow is the \(\mathbb{R}^*\)-principal bundle map \(p\) from \(\mathcal{M}_\omega\) to \(\mathcal{M}_\Omega\) described in Section 4.1; the last up and down horizontal arrows are the identifications described in Remark 8.1.2 and \(\hat{h}\) can be viewed as a principal bundle map along the group homomorphism \(\lambda \mapsto \log |\lambda|\) from \(\mathbb{C}^*\) to \(\mathbb{R}\).

We now consider the real function \(F_L\) defined on \(\mathcal{B}(E)\) by

\[
F_L = \mathbb{I}_\omega \circ \hat{h},
\]
and we define \(\theta_L\) — a real 1-form on \(\mathcal{B}(E)\) — by

\[
\theta_L = -(\hat{h}^*\tau) \circ J_{\mathcal{B}(E)} = d^c(\mathbb{I}_\omega \circ \hat{h}) = d^c F_L,
\]
where \(d^c\) is relative to the natural complex structure, \(J_{\mathcal{B}(E)}\), of \(\mathcal{B}(E)\). Denote by \(\omega_L\) the real 2-form on \(\mathcal{B}(E)\) defined by

\[
\omega_L = d\theta_L = dd^c(\mathbb{I}_\omega \circ \hat{h}) = dd^c F_L.
\]

**Theorem 8.6.1.** (i) The 2-form \(\omega_L = dd^c F_L\) defined by (8.6.5) descends to a real 2-form, \(\tilde{\omega}_L\), on \(\mathcal{B}_0(E) = H_L(M)\backslash \mathcal{B}_0(E)\), which is there the Kähler form of a \(SU(N + 1)\)-invariant Kähler structure compatible with the natural complex structure, \(J_{\mathcal{B}_0(E)}\), of \(\mathcal{B}_0(E)\).

(ii) With respect to this symplectic structure, \(\mu_L\) is the momentum of the induced action of \(SU(N + 1)\).

**Proof.** At each point \(s\) of \(\mathcal{B}(E)\), the tangent space \(T_s \mathcal{B}(E)\) is naturally identified with the linear Lie algebra \(\mathfrak{gl}(N + 1, \mathbb{C})\): each \(A\) in \(\mathfrak{gl}(N + 1, \mathbb{C})\) is identified with the vector in \(T_s \mathcal{B}(E)\) tangent to the curve \(s_t = s \exp(tA)\) in \(\mathcal{B}(E)\) at \(t = 0\). In the sequel, we denote by \(A^+,\) resp. \(A^-\), the hermitian, resp. anti-hermitian, part of \(A\). For any \(h\) in \(\mathcal{H}^{>0}(L)\), the tangent space \(T_h \mathcal{H}^{>0}(L)\) is naturally identified with the space of real functions on \(M\): each real function \(f\) is then determined by the curve \(h_t = e^{-2t} f h\), cf. Remark 8.1.2. Theorem 8.6.1 is then a consequence of the following series of lemmas.

**Lemma 8.6.1.** The differential of \(\hat{h}\) at \(s\) is given by

\[
(\hat{h}_s)_s(A) = \sum_{i,j=0}^N A^+_{ij}(s_i, s_j) h_0 = \mu_s^{\mathbb{A}^+}
\]
for any \(A\) in \(\mathfrak{gl}(N + 1, \mathbb{C})\), by setting \(\mu_s^{\mathbb{A}^+} := \mu_{\mathbb{F}S}^{\mathbb{A}^+} \circ j_s\), where, we recall, \(\mu_{\mathbb{F}S}^{\mathbb{A}^+}\) denotes the momentum of the induced vector field \(\mathbb{A}^+\) on the standard projective space \(\mathbb{P}^N\).
Proof. With the above convention we have that
\[
(\hat{h}_s)(A) = \frac{d}{dt}_{t=0} \frac{1}{2} \log \sum_{j=0}^{N} |(s \cdot \exp(tA))_j|^2_{h_0}
\]
(8.6.7)
\[
= \frac{\Re(\sum_{i,j=0}^{N} A_{ij} \langle s_i, s_j \rangle_{h_0})}{\sum_{k=0}^{N} |s_k|^2_{h_0}}
= \frac{\sum_{i,j=0}^{N} A_{ij}^* \langle s_i, s_j \rangle_{h_0}}{\sum_{k=0}^{N} |s_k|^2_{h_0}}.
\]
The second equality in (8.6.6) readily follows from the expression (8.3.14) of \( \varphi_L \) and the expression (6.3.11) of the momentum of \( \hat{\alpha} \), for \( a = i \hat{A}^+ \) (we here admit the case when \( A = \lambda I \), for some complex number \( \lambda \); then, \( \hat{\alpha} = 0 \) and the “momentum” \( \mu_{\hat{\alpha}} \) is the constant function \( \Re(\lambda) \)).

Lemma 8.6.2. For any \( s \) in \( \mathcal{B}(E) \) and any \( A \) in \( \mathfrak{gl}(N+1, \mathbb{C}) = T_s \mathcal{B}(E) \), we have
\[
\theta_L(A) = \frac{1}{V} \int_M \mu_{\hat{\alpha}}^* v_{M,s},
\]
(8.6.8)
with \( \mu_{\hat{\alpha}}^* = \mu_{\hat{\alpha}}^\mathcal{M} \circ j_s \).

Proof. From (8.6.4), we infer \( \theta_L(A) = d^c(1 \circ \hat{h})(A) = -d(1 \circ h)(iA) = -\tau(\langle \hat{h}_s \rangle_s(iA)) \), where, we recall, \( \tau \) is defined by (4.1.2). By Lemma 8.6.1, we thus get: \( \theta_L(A) = -\frac{1}{V} \int_M \mu_{\hat{\alpha}}^*(iA)^+ v_{M,s} = \frac{1}{V} \int_M \mu_{\hat{\alpha}}^* v_{M,s} \).

Lemma 8.6.3. For any \( a \) in \( \mathfrak{u}(N+1) \) and any \( B \) in \( \mathfrak{gl}(N+1, \mathbb{C}) = T_s \mathcal{B}(E) \), we have that
\[
\frac{d}{dt}_{|t=0} \mu_{\exp(tB)} = \omega_{\mathcal{M}}(\hat{B}, \hat{\alpha}) \circ j_s.
\]
(8.6.9)
Proof. Since \( \mu_{\hat{\alpha}}^* = \mu^\mathcal{M} \circ j_s \circ \varphi_L \), we have \( \frac{d}{dt}_{|t=0} \mu_{\exp(tB)} = d \mu^\mathcal{M}(\hat{B}) \circ j_s = \omega_{\mathcal{M}}(\hat{B}, \hat{\alpha}) \circ j_s \).

In the following lemma, for any \( A \) in \( \mathfrak{gl}(N+1, \mathbb{C}) \) we set \( \hat{A}^\dagger = \hat{A} - \hat{A}_{|M} \), where \( \hat{A}_{|M} \) denotes the orthogonal projection on TM with respect to the Fubini-Study metric of \( \mathbb{P}^N \), and we write \( \omega_{\mathcal{M}}(\hat{A}^\perp, \hat{B}^\perp)|_M \) instead of \( \omega_{\mathcal{M}}(\hat{A}^\dagger, \hat{B}^\dagger) \circ j_s \).

Lemma 8.6.4. For any \( s \) in \( \mathcal{B}(E) \) and any \( A, B \) in \( \mathfrak{gl}(N+1, \mathbb{C}) = T_s \mathcal{B}(E) \), we have that
\[
d\theta_L(A, B) = \frac{1}{V} \int_M \left( \omega_{\mathcal{M}}(\hat{A}^\perp, \hat{B}^\perp)|_M \right) v_{M,s}.
\]
(8.6.10)
Proof. We use the general formula:
\[
d\theta_L(A, B) = A \cdot (\theta_L(B)) - B \cdot (\theta_L(A)) - \theta_L([A, B])
\]
(8.6.11)
where \( A, B \) are viewed as vector fields on \( \mathcal{B}(E) \) via the overall identification \( \mathfrak{gl}(N+1, \mathbb{C}) = T_s \mathcal{B}(E) \) for any \( s \) in \( \mathcal{B}(E) \) (these vector fields must not be
confused with the induced vector field on \( \mathbb{P}^m \), denoted \( \hat{A}, \hat{B} \). By (8.6.9) we have
\[
A \cdot (\mu^B_M) = \omega_{FS}(\hat{A}, \hat{B})|_M,
\]
whereas, by (3.2.4) in Lemma 3.2.1, with \( \hat{\phi}_t = \frac{1}{2} \log \sum_{j=0}^N |(s \cdot \exp(tA))|_{h_0}^2 \), and by using (8.6.7), we get
\[
A \cdot v_{M,s} = -\Delta_{M,s}(\mu^A_M) v_{M,s}.
\]
We thus obtain
\[
A \cdot (\theta_L(B)) = \frac{1}{V} \int_M (\omega_{FS}(\hat{A}, \hat{B})|_M - \mu^B_M \Delta_{M,s}(\mu^A_M)) v_{M,s}.
\]
\[
= \frac{1}{V} \int_M (\omega_{FS}(\hat{A}, \hat{B})|_M - (d\mu^A_M, d\mu^B_M)|_{M,s} v_{M,s}.
\]
\[
= \frac{1}{V} \int_M ((\omega_{FS}(\hat{A}, \hat{B})|_M - \omega_{M,s}(\hat{A}^-|_M, \hat{B}^-|_M)) v_{M,s}.
\]
Observe that
\[
\int_M \omega_{M,s}(\hat{A}^-|_M, \hat{B}^-|_M) v_{M,s} = 0,
\]
for any \( A, B \) in \( \mathfrak{gl}(N+1, \mathbb{C}) \), as \( \hat{A}^-|_M \) and \( \hat{B}^-|_M \) are both hamiltonian vector fields on \( M \) with respect to \( \omega_{M,s} \). In general, the orthogonal projection of the gradient of a function on the tangent bundle of a submanifold is the gradient of the restriction of the function with respect to the induced metric, so that
\[
\hat{i}A^+|_M = \text{grad}_{\omega_{M,s}}(\mu^A_M),
\]
and similarly for \( \hat{B}^-|_M \). Finally, since \( [A, B]^+ = [A^-, B^-] + [A^+, B^+] = [A^-, B^-] - [iA^+, iB^+] \), we get:
\[
\theta_L([A, B]) = \frac{1}{V} \int_M (\omega_{FS}(\hat{A}^-|_M, \hat{B}^-|_M) - \omega_{FS}(\hat{A}^+|_M, \hat{B}^+|_M)) v_{M,s}.
\]
By plugging (8.6.14) and (8.6.17) into (8.6.11) and by using (8.6.15), we easily get (8.6.10).

**Lemma 8.6.5.** The 2-form \( d\theta_L \) descends to a \( U(N+1) \)-invariant Kähler form on \( \mathcal{B}(E) \).

**Proof.** Since any two bases \( s, s' \) induce the same hermitian inner product on \( E \), as soon as \( \gamma \) belongs to \( U(N+1) \), the map \( h \) from \( \mathcal{B}(E) \) to \( \mathcal{H}^{>0}(L) \) is \( U(N+1) \)-invariant, hence also \( F_L, \theta_L \) and \( d\theta_L \). On the other hand, from (8.6.10), we readily infer that \( d\theta_L(A, B) = 0 \), whenever \( A \) is a multiple of the identity, so that \( \hat{A} = 0 \), or \( A \) belong to the Lie algebra of \( \mathcal{H}_L(M) \), cf. Proposition 8.3.2; since \( d\theta_L \) is (evidently) closed, it follows that \( d\theta_L \) is invariant under the (left) action of \( \mathcal{H}_L(M, J) \) and the (right) action of \( \mathbb{C}^* \), and actually descends to a 2-form \( \hat{\omega}_L \) on \( \mathcal{B}_0(E) = \mathcal{H}_L(M) \backslash \mathcal{B}(E)/\mathbb{C}^* \). On \( \mathcal{B}_0(E), \theta_L \) is clearly \( J_{\mathcal{B}_0(E)} \)-invariant and the corresponding symmetric form
is $g_L(A, B) = \frac{1}{V} \int_M ((g_{FS}(\hat{A}^\perp, \hat{B}^\perp))_{|M}) v_{M,s}$, which descends to a positive definite metric on $\mathcal{B}_0(E)$.

This proves the first assertion of Theorem 8.6.1. To prove the second assertion, we use the fact that $\theta_L$ is invariant under the action of $SU(N+1)$ on $\mathcal{B}_0(E)$: it follows that for any $a$ in $\mathfrak{su}(N+1)$, the “momentum” of the induced vector field $\hat{a}$ with respect to $d\theta_L$ is simply $\theta_L(\hat{a})$, cf. Remark 9.6.2. By (8.6.8), $\theta_L(\hat{a})$ is equal to $(\mu_L(s), a)$. Since $\mu_L$ is $H_L(M)$-invariant, cf. Lemma 8.4.2, we infer that it is the momentum of $\hat{a}$ on $\mathcal{B}_0(E)$ with respect to the induced symplectic form $\tilde{\omega}_L$.

\[ \square \]

7. Balanced metrics

**Definition 15.** A pair $(M, L)$ formed of a (compact) projective manifold $M$ and a very ample holomorphic line bundle $L$ over $M$ is balanced if there exists a basis $s$ of $E = H^0(M, L)$ such that $\mu_L(s) = 0$. Then, $(M, L)$ is said to be balanced with respect to $s$.

Equivalently, $(M, L)$ is balanced with respect to $s$ if $s$ is orthonormal, up to normalization, with respect to the hermitian inner product $\langle \cdot, \cdot \rangle_{h(s)}$ on $H^0(M, L)$ determined by the induced pointwise hermitian product $h_s$, as defined in (8.4.8).

**Remark 8.7.1.** It readily follows from Lemma 8.4.1 that $(M, L)$ is balanced with respect to $s$ if and only if the corresponding Bergman density $B_{L, h_s}$ — cf. Section 8.2 — is constant. This constant is then equal to $\frac{N+1}{V}$, where, we recall, $N+1$ stands for the (complex) dimension of $H^0(M, L)$ and $V$ denotes the total volume determined by the Kähler class $2\pi c_1(L)$.

**Proposition 8.7.1.** If $(M, L)$ is balanced with respect to a basis $s$ of $E = H^0(M, L)$, then $s$ is uniquely determined up to the actions of $\mathbb{C}^*$, $SU(N+1)$ and $H_L(M)$.

**Proof.** By Lemma 8.4.2, if $(M, L)$ is balanced with respect to $s$ it is also balanced with respect to any other basis deduced from $s$ by the action of $\mathbb{C}^*$, $SU(N+1)$ and $H_L(M)$. It remains to check that $(M, L)$ is not balanced with respect to $s \cdot \gamma$, whenever $\gamma$ is an element of $SL(N+1, \mathbb{C})$ not in $SU(N+1)$ (it is in fact sufficient to assume that $\gamma = \exp(itu)$, when $a$ is in $\mathfrak{su}(N+1)$ but $\hat{a}$ is not in $\mathfrak{h}_L$). This is a direct consequence of $\mu_L$ being the momentum map of the $SU(N+1)$-action on $\mathcal{B}_0(E)/H_L(M)$ with respect to the symplectic form $d\theta_L = \omega_L$ given by (8.6.10). Indeed, for any $a$ as above, we have that

\begin{equation}
(8.7.1) \quad \frac{d}{dt}|\mu^a_L(s \cdot \exp(itu))|^2 = \frac{2}{V} \int_M g_{FS}(\hat{a}^\perp, \hat{a}^\perp) v_{M,s \cdot \exp(itu)}
\end{equation}

which is positive for any real value of $t$, cf. [30] and Remark 9.6.2 below. It follows that $\mu_L(s \cdot \exp(itu))$ is different from 0 except for $t = 0$.

**Remark 8.7.2.** In [76], S. K. Donaldson considers the space, $\text{Herm}^+(E)$ of (positive definite) hermitian inner product on $E$ — denoted by $M$ in [76]
— which is naturally identified with the quotient $\mathcal{B}(E)/U(N+1)$, and the maps $\operatorname{Hilb}$ from $H^*(L)$ to $\mathcal{H}(E)$ and $\operatorname{FS}$ from $\mathcal{H}(E)$ to $H^*(L)$, defined by

$$
(8.7.2) \quad \operatorname{Hilb}(h) = \frac{N + 1}{V} \langle \cdot, \cdot \rangle_h, \quad \operatorname{FS}(H) = h_s,
$$

for any $h$ in $H^*(L)$ and any $H = [s]$ in $\operatorname{Herm}^+(E) = \mathcal{B}(E)/U(N + 1)$. We thus get the following diagram, where the map $H$ from $\mathcal{B}(E)$ to $\operatorname{Herm}^+(E)$ is defined by $H(s) = [s]$ (here, and above, $[s]$ denotes the class of $s$ mod $U(N + 1)$, identified with the unique element of $\operatorname{Herm}^+(E)$ with respect to which $s$ is orthonormal):

$$
(8.7.3) \quad H^*(L) \xrightarrow{\operatorname{Hilb}} \operatorname{Herm}^+(E) \xrightarrow{\operatorname{FS}} \mathcal{B}(E)
$$

As noticed by Donaldson, $(M, L)$ is balanced with respect to $s$ if and only if $\operatorname{FS} \circ \operatorname{Hilb}(h_s) = h_s$ or, equivalently, $\operatorname{Hilb} \circ \operatorname{FS}([s]) = [s]$.

It has been observed by S. Zhang [201] and H. Luo [139] that the concept of balanced pair $(M, L)$ is closely connected to the concept of Mumford-Chow stability of $M$ as a submanifold of $\mathbb{P}(H^0(M, L)^*)$. This in turn is conjectured to be closely connected to the existence of Kähler metrics with constant scalar curvature or, more generally, with extremal Kähler metrics. A strong evidence is then given by the following theorem of S. Donaldson:

**Theorem 8.7.1 (S. Donaldson [75] Theorems 1, 2, 3).** Let $M$ be a compact complex manifold and let $L$ be an ample holomorphic line bundle over $M$. Suppose that $L^k$ is very ample for $k \geq k_0$ and, for any $k \geq k_0$, denote by $E_k = H^0(M, L^k)$ the space of holomorphic sections of $L^k$ and by $N_k + 1$ the dimension of $E_k$. Assume moreover that $H(M, L) = H_0(M, J) = \{1\}$. Then:

(i) If the pair $(M, L^k)$ is balanced, for some $k \geq k_0$, then it is balanced with respect to a uniquely defined element of $\operatorname{FS} E_k$.

(ii) Suppose that, for each sufficiently large positive integer $k$, $(M, L^k)$ is balanced with respect to some basis $s^{(k)}$ of $H^0(M, L^k)$ and that the corresponding normalized induced Fubini-Study metrics $\omega_{FS,k} := \frac{1}{k} \omega_{M,s^{(k)}}$ converge in $C^\infty$ to some limit $\omega_\infty$ as $k$ tends to $+\infty$. Then, $\omega_\infty$ is of constant scalar curvature.

(iii) Suppose that $(M, J)$ admits a Kähler metric $\omega$ of constant scalar curvature polarized by $L$. Then, there exists $k_0 > 0$ such that for any $k \geq k_0$ the pair $(M, L^k)$ is balanced with respect to some basis $s^{(k)}$ of $H^0(M, L^k)$ — essentially unique by (i) — and the corresponding sequence of renormalized metrics $\omega_{FS,k} := \frac{1}{k} \omega_{M,s^{(k)}}$ then $C^\infty$ converges to $\omega$.

**Proof.** Part (i) follows from Proposition 8.7.1. Part (ii) can be obtained by applying Theorem 8.2.1 to the family of Kähler forms $\omega_{FS,k} = \frac{1}{k} \omega_{M,s^{(k)}}$, which we simply denote $\tilde{\omega}_k$, $k \geq k_0$; the corresponding metric is similarly denoted by $\tilde{g}_k$. Then, by Lemma 8.4.1, $\tilde{\omega}_k$ is the Kähler form of $M$ when $M$ is polarized by $L$, equipped with the pointwise hermitian
product of $L$, say $h_k$, induced by $h(s^{(k)})$: for each $u$ in $L$, we then have $|u|^2_{h_k} := |u^{(k)}|^2$. For each $k \geq k_0$, the estimate (8.2.5) for $q = 1$ now reads

\begin{equation}
|B_{L^p, h_k^{(p)}}(x) - \frac{1}{(2\pi)^m} \frac{1}{4p} s^4_{\omega_k} | C^{(k)} \leq \frac{C_{1,\ell}}{k},
\end{equation}

where $h_k^{(p)}$ denotes the induced hermitian inner product on $L^p$. The main point here is that the constants $C_{1,\ell}$ are independent of $k$, as the $\omega_k$ are $C^\infty$ bounded by assumption. Now, by assumption $B_{L^p, h_k^{(k)}} = B_{L^p, h_l^{(k)}}$ is constant, hence equal to $\frac{d(L^k)}{V_k}$, with $V_k = k^m V$; for $p = k$, (8.7.4) then reads

\begin{equation}
\left| k \left( \frac{d(L^k)}{V_k} - \frac{1}{(2\pi)^m} \frac{1}{4} s^4_{\omega_k} \right) \right| C^{(k)} \leq \frac{C_{1,\ell}}{k},
\end{equation}

for each $k \geq k_0$. On the other hand, from Proposition 8.2.1, we infer that

\begin{equation}
\frac{d(L^k)}{V_k} = \frac{d(L^k)}{k^m} \frac{d(L^k)}{V} = \frac{1}{(2\pi)^m} + \frac{k^{-1}}{(2\pi)^m} \int_M \frac{s}{4} v + O(k^{-2}).
\end{equation}

We infer that for each $\ell$ and each $k \geq k_0$, we have that

\begin{equation}
\left| s^4_{\omega_k} - \frac{\int_M s v}{V} \right|^2 | C^{(k)} \leq \frac{C_{1,\ell}}{k},
\end{equation}

for some positive constant $C_{1,\ell}$ independent of $k$. It follows that $s^4_{\omega_k} C^\infty$ converges to the constant $\int_M s v / V$, which is then equal to $s g_{\infty}$ (note that the mean value of the scalar curvature of all metrics of Kähler forms $\omega_k$ are the same, as well as the scalar curvature of $g_{\infty}$, as they all have the same Kähler class $2\pi c_1(L)$). This completes the proof of Part (ii) of Theorem 8.7.1 (more details in [75]). Part (iii) is the hard core of Theorem 8.7.1 and its proof goes beyond the scope of these notes: the reader is referred to Donaldson’s original paper [75]; cf. also the Bourbaki Seminar [30] by O. Biquard. □

**Remark 8.7.3.** Donaldon’s theorem has been extended by T. Mabuchi to include general extremal polarized Kähler metrics and any group $H_L(M)$. Again, the proof of Mabuchi’s theorem lies beyond the scope of these notes and the reader is referred to original papers [143], [144], [145], [146].

### 8.8. $S^1$-actions on polarized Kähler manifold

Let $M$ be a connected smooth manifold and $L$ a hermitian complex line bundle over $M$, equipped with a $C$-linear connection $\nabla$ which preserves the hermitian inner product. The curvature $R^\nabla$ of $\nabla$ is of the form $R^\nabla = i \rho^\nabla$, where $\rho^\nabla$ — the curvature form — is a closed real 2-form, cf. Section 1.19. Suppose that $M$ admits a smooth (left) action of $S^1$ which preserves $\rho^\nabla$; equivalently, $\mathcal{L}_X \rho^\nabla = d(\iota_X \rho^\nabla) = 0$, where $X$ denotes the generator of the $S^1$-action, i.e. the (real) vector field on $M$ defined by

\begin{equation}
X(x) = \frac{d}{dt}_{|t=0} e^{it} \cdot x,
\end{equation}

for any $x$ in $M$. We say that this action is $\rho^\nabla$-hamiltonian if there exists a smooth real function $h^X$ on $M$ such that $\iota_X \rho^\nabla = -dh^X$; the $\rho^\nabla$-momentum
$h^X$ is then well defined up to an additive constant (notice however that we don’t assume that $\rho^\nabla$ be a symplectic form).

For any choice of $h^X$, we consider the vector field $\hat{X}$ on (the total space of) $L$ defined by

$$\hat{X} = \hat{X} - h^X T,$$

where $\hat{X}$ denotes the horizontal lift of $X$ with respect to $\nabla$ — cf. Section 1.6 — and $T$ the generator of the natural action of $S^1$ on $L$. We then have

**Proposition 8.8.1.** Any $\rho^\nabla$-hamiltonian action of $S^1$ on $M$ can be lifted into an action of $S^1$ on $L$ which preserves the hermitian inner product of $L$ and the horizontal distribution $H^\nabla$ of $\nabla$. For any such lifted action, the generator is given by (8.8.2) for an appropriate choice of the $\rho^\nabla$-momentum $h^X$, uniquely defined up to an additive integer.

**Proof.** For any $S^1$-action on $M$, the generator, $X$, of this action can be lifted into a real vector field on $L$ and any lift of $X$ has the form $\hat{X} = f_1 T - f_2 iT$, where $f_1, f_2$ are any real functions defined on $M$, regarded as defined on $L$ by pullback. Since $\nabla$ is a hermitian connection, $\hat{X}$ certainly preserves the hermitian inner product of $L$, and so does $T$, whereas $iT$ does not; it follows that a lift of $X$ on $L$ preserves the hermitian inner product of $L$ if and only if it has the form (8.8.2) for any real function $h^X$ defined on $M$. Note that for any such function, $\hat{X}$ commutes with $T$: this is because $T$ preserves $\nabla$ — in fact any $C$-linear connection — and $dh^X(T)$ is clearly zero. Moreover, $\hat{X}$ preserves $\nabla$, i.e. the horizontal distribution $H^\nabla$, if and only if $h^X$ is a $\rho^\nabla$-momentum: indeed, for any vector field $Z$ on $M$, we have that

$$[\hat{X}, \hat{Z}] = [\hat{X}, \hat{Z}] - [dh^X T, \hat{Z}]$$

(8.8.3)

$$= [X, Z] + (\rho^\nabla(X, Z) + dh^X(Z)) T,$$

where $[\hat{X}, \hat{Z}]$ is the horizontal lift of the bracket $[X, Z]$; the latter expression is horizontal, then equal to $\hat{[X, Z]}$, for any $Z$ — meaning that $\hat{X}$ preserves $H^\nabla$ — if and only if the coefficient of $T$ is identically zero, meaning that $h^X$ is a $\rho^\nabla$-momentum.

We now show that $\hat{X}$ can be made the generator of a lifted action of $S^1$ on $L$ by adding an appropriate real constant to $h^X$, well defined up to an integer. Denote by $\Phi^X_t$ the flow of $X$ on $M$ and by $\Phi^X_{\hat{t}}$ the flow of $\hat{X}$ on $L$; then, the two flows are related by $\pi \circ \Phi^X_t = \Phi^X_{\hat{t}} \circ \pi$ for any value of $t$, where $\pi$ denotes the projection from $L$ to $M$. Moreover, since the two summands, $\hat{X}$ and $-h^X T$ of $\hat{X}$ commute — this is because $dh^X(X) = 0$ — the flow $\Phi^X_{\hat{t}}$ is the product of the flows of $\hat{X}$ and $-h^X T$ which both preserve the hermitian inner product of $L$; since $\Phi^X_{\hat{2t}}(x) = x$ for any $x$ in $M$, it follows that $\Phi^X_{\hat{2t}}(\xi) = \zeta(x) \xi$ for any $\xi$ in $L_x$, for some $\zeta(x)$ in $S^1$. We now claim that $\zeta$ is constant. Indeed, for any vector field $Z$ on $M$, the derivative of $\zeta$ along $Z$ is determined by $\Phi^X_\hat{t} \cdot \hat{Z} = \hat{Z} - id\zeta(Z) T$, where $\hat{Z}$ denotes the horizontal lift of $Z$ on $L$. Since $\Phi^X_t$ preserves the horizontal distribution for all values of $t$, in particular for $t = 2\pi$, $\Phi^X_\hat{2t} \cdot \hat{Z}$ must be horizontal and $d\zeta(Z)$ is then
equal to zero. It follows that \( \zeta = e^{2\pi c} \), for some real constant \( c \) which we can choose in the interval \([0, 2\pi)\) and is then uniquely defined. If we substitute \( h^X + c \) to \( h^X \) and if we still denote the corresponding lift of \( X \) by \( \tilde{X} \), we then have \( \Phi_{2\pi}(\xi) = \xi \) for any \( \xi \) in \( L \). This means that \( e^{it} \cdot \xi := \Phi_t(\xi) \) is a \( S^1 \)-action on \( L \), which clearly lifts the action of \( S^1 \) on \( M \).

**Remark 8.8.1.** To any action of \( S^1 \) which preserves \( \rho^\nabla \) as above but a priori is not assumed to be \( \rho^\nabla \)-hamiltonian, we associate the map \( \zeta : M \to S^1 \) defined by \( \Phi_{2\pi}(\xi) = \zeta(x) \xi \) for any \( \xi \) in \( L_x \): \( \zeta \) is then the holonomy of \( \nabla \) along the loop \( \Phi_t^X(x) \). The above argument can then be interpreted as follows: The action of \( S^1 \) is \( \rho^\nabla \)-hamiltonian if and only if there exists a smooth real function, \( h^X \), which integrates \( \zeta \), i.e. such that \( \zeta(x) = e^{2\pi ih^X(x)} \), and \( h^X \) is then a \( \rho^\nabla \)-momentum of this action.

This action of \( S^1 \) on \( L \) induces a linear (left) action of \( S^1 \) on the space \( \Gamma(L) \) of smooth sections of \( L \), defined as follows:

\[
(8.8.4) \quad (e^{it} \cdot s)(x) = \Phi_t^X(s(\Phi_t^X(x))).
\]

In analogy with the usual Lie derivative — cf. Section 1.4 — the opposite of the derivative of this action can be regarded as a Lie derivative along \( X \) operating on the space \( \Gamma(L) \), denoted by \( \mathcal{L}_X \), with

\[
(8.8.5) \quad \mathcal{L}_X s := -\frac{d}{dt}_{|t=0} (e^{it} \cdot s).
\]

We then have:

**Proposition 8.8.2.** For any \( s \) in \( \Gamma(L) \), the Lie derivative \( \mathcal{L}_X s \) has the following expression

\[
(8.8.6) \quad \mathcal{L}_X s = \nabla_X s + ih^X s.
\]

**Proof.** This is a simple computation: for any \( x \) in \( M \),

\[
(\mathcal{L}_X s)(x) = -\frac{d}{dt}_{|t=0} \Phi_t^X(s(\Phi_t^X(x)))
= -\tilde{X}(s(x)) + s(\tilde{X}(x))
= -\tilde{X}(s(x)) + h^X(x) T(s(x)) + \tilde{X}(s(x)) + (\nabla_X s)(x)
= (\nabla_X s)(x) + ih^X(x)s(x),
\]

where we used (1.6.10) of Section 1.6.

As an illustration of Proposition 8.8.1, we consider the case when \( L = K^*(M) \) is the anti-canonical line bundle of a Kähler manifold \((M, g, J, \omega)\). Recall that the Kähler structure of \( M \) induces a hermitian inner product on \( K^*(M) \), whose curvature form is the Ricci form \( \rho \), cf. Section 1.19. We then have

**Proposition 8.8.3.** Consider any action of \( S^1 \) on \( M \), of generator \( X \), which preserves the whole Kähler structure \((g, J, \omega)\) and which is hamiltonian with respect to \( \omega \), with momentum \( h^X \). Then, this action also preserves the Ricci form \( \rho \) and is \( \rho \)-hamiltonian, of \( \rho \)-momentum \( \frac{1}{2} \Delta h^X \).
Moreover, this action lifts canonically as a holomorphic action on the anti-canonical line bundle $K_M^{-1}$, with generator $\bar{X} = \bar{X} - \frac{1}{2}\Delta h^X T$, and the induced linear action on $\Gamma(K_M^{-1})$ coincides with the usual Lie derivative.

**Proof.** Since $S^1$ preserves the metric and the complex structure, it also preserves the Ricci form $\rho$; moreover, it readily follows from Lemma 1.23.4 that $X$ is also $\rho$-hamiltonian, of $\rho$-momentum equal to $\frac{1}{2}\Delta h^X$, up to an additive constant. If $\Phi_t^X$ denotes the flow of $X$, so that the $S^1$-action on $M$ is given by $e^{it} \cdot x = \Phi_t^X(x)$, the canonical lifted action on $K^*(M)$ is simply given by $e^{it} \cdot \Psi := (\Phi_t^X)_* (\Psi)$, for any $\Psi$ in $K^*(M)$. This makes sense, as $\Phi_t^X$ is a biholomorphic transformation of $M$ for each $t$.

The induced linear action on $\Gamma(K^*(M))$ is then the usual Lie derivative, as defined in Section 1.4. Now, the Lie derivative $\mathcal{L}_{\bar{X}}$ and the Levi-Civita covariant derivative $D$ acting on sections of $T^{1,0}M \cong TM$ are related by $\mathcal{L}_{\bar{X}} = DX - DX$, cf. Section 1.23; the operator $\mathcal{L}_{\bar{X}} - DX$, acting on sections of $K^*(M)$ then coincides with the induced action of the endomorphism $DX$ on elements of $K^*(M)$, which is exactly $\Delta^m(T^{1,0}M)$. This action can be easily made explicit as follows. Consider any point $x$ of $M$ and any element $\Psi$ in $K_x^*(M)$, which we can choose of the form $\Psi = \varepsilon_1 \wedge \ldots \wedge \varepsilon_m$, where the $\varepsilon_j$, $j = 1, \ldots, m$, form a basis of $T_x^1$; we can even assume that this basis is hermitian-orthonormal, meaning that $\varepsilon_j = \frac{1}{\sqrt{2}}(\varepsilon_j - iJ\varepsilon_j)$, where $e_1, Je_1, \ldots, e_m, Je_m$ is a $g$-orthonormal basis of $T_xM$; we then have

\begin{equation}
(\mathcal{L}_{\bar{X}} - DX)(\varepsilon_1 \wedge \ldots \wedge \varepsilon_m) = -\sum_{j=1}^m \varepsilon_1 \wedge \ldots \wedge D_{\varepsilon_j}X \wedge \ldots \wedge \varepsilon_m,
\end{equation}

where the rhs is equal to $-\sum_{j=1}^m (D_{\varepsilon_j}X, \varepsilon_j) \varepsilon_1 \wedge \ldots \wedge \varepsilon_m$, where the bracket $(\cdot, \cdot)$ here just mean the $\mathbb{C}$-bilinear extension of $g$.

Now, $-\sum_{j=1}^m (D_{\varepsilon_j}X, \varepsilon_j)$ is equal to $-\frac{1}{2}\sum_{j=1}^m (D_{e_j-iJe_j}X, e_j + iJe_j)$, hence to $\frac{1}{2}\delta X^\flat - \frac{i}{2}\delta(JX^\flat)$ (this is because $D_{Je_j}X = JDX_{e_j}$, as $X$ is (real) holomorphic); finally, $\delta X^\flat = 0$, as $X$ is Killing, whereas $\delta(JX^\flat) = -\Delta h^X$, since $X^\flat = d^c h^X$. We then get

\begin{equation}
\mathcal{L}_{\bar{X}} \Psi = DX \Psi + \frac{i}{2} \Delta h^X \Psi,
\end{equation}

for any section $\Psi$ of $K^*(M)$. By comparing with (8.8.6), we conclude that the choice of the $\rho$-momentum map corresponding the canonical lift of the $S^1$-action on $K^*(M)$ — meaning that its generator is $\bar{X} = \bar{X} - \frac{1}{2}\Delta h^X T$ — is exactly $\frac{1}{2}\Delta h^X$, i.e. the $\rho$-momentum map of $X$ which integrates to 0.

**Remark 8.8.2.** From Lemma 1.23.4 we immediately infer that $i_X \rho = -d(\frac{1}{2}\Delta f)$ — hence that $\frac{1}{2}\Delta f$ is a $\rho$-momentum of $X$ — for any hamiltonian Killing vector field $X = J\text{grad}_g f$ defined on any compact Kähler manifold. The new information given by Proposition 8.8.3 is that if $X$ is the generator of a $S^1$-action on $M$, then $\frac{1}{2}\Delta f$ is exactly the $K^*(M)$-momentum corresponding to the canonical lift of this action on $K^*(M)$.

We now consider the case when the pair $(M, L)$ is a polarized Kähler manifold as defined in Section 8.1, so that the curvature form $\rho^\nabla$ is the Kähler form $\omega$. Propositions 8.8.1 and 8.8.2 can then be specialized as follows
Proposition 8.8.4. Let \((M, L)\) be a polarized Kähler manifold and consider an action of \(S^1\) on \(M\) which preserves the whole Kähler structure \((g, J, \omega)\) and is hamiltonian with respect to \(\omega\). Then, for an appropriate choice of the momentum \(h^X\), uniquely determined up to an additive integer, the lift \(\hat{X} = X - h^X T\) is the generator of a holomorphic action of \(S^1\) on \(L\).

Moreover, the induced linear action on \(\Gamma(L)\) given by (8.8.6) preserves the subspace \(H^0(M, L)\) of holomorphic section.

Proof. For any choice of \(h^X\), we already showed in the proof of Proposition 8.1.2 that \(\hat{X}\) is a (real) holomorphic vector field on \(L\). When \(h^X\) is so chosen that \(\hat{X}\) is the generator of a \(S^1\)-action — cf. Proposition 8.8.1 — this action is then holomorphic. This proves the first assertion.

A section \(s\) of \(L\) is holomorphic if and only if \(\hat{\partial}s = 0\), where the Cauchy-Riemann operator of the holomorphic structure is related to the Chern connection \(\nabla\) by \(\hat{\partial} = \nabla^{0, 1}\), cf. Section 1.6; equivalently, a section \(s\) of \(L\) is holomorphic if and only if

\[
\nabla_{JZ}s = i\nabla_Zs, 
\]

for any (real) vector field \(Z\) on \(M\). For any vector field \(Z\) on \(M\) and for any smooth section \(s\) of \(L\), we have:

\[
\nabla_Z(\mathcal{L}_X s) = \nabla_Z(\nabla_X s) + idh^X(Z)s + ih^X\nabla_Zs \\
= \nabla_X(\nabla_Z s) + R^Z_{X,Z}s - \nabla_{[X,Z]} s + idh^X(Z)s + ih^X\nabla_Zs \\
= i(\omega(X, Z) + dh^X(Z))s \\
+ \nabla_X(\nabla_Z s) - \nabla_{[X,Z]} s + ih^X\nabla_Zs \\
= \nabla_X(\nabla_Z s) - \nabla_{[X,Z]} s + ih^X\nabla_Zs.
\]

Now, assume that \(s\) is holomorphic and substitute \(JZ\) to \(Z\) in the last term of the rhs: we then get \(\nabla_{JZ}s = i\nabla_Zs\), whereas \(\nabla_{[X,JZ]}s\) is equal to \(\nabla_{J[X,Z]}s\), as \(X\) is (real) holomorphic, hence to \(i\nabla_{[X,Z]}s\) as \(s\) is holomorphic; we infer \(\nabla_{JZ}(\mathcal{L}_X s) = i\nabla_Z(\mathcal{L}_X s)\); by (8.8.9) this means that \(\mathcal{L}_X s\) is a holomorphic section. \(\square\)

Remark 8.8.3. The momentum \(h^X\) appearing in Proposition 8.8.4 will be called a \(L\)-momentum of \(X\): \(L\)-momenta of \(X\) are then well defined up to an additive integer and \(\int_M h^X v_g\) is then well defined as an element of \(\mathbb{R}/\mathbb{Z}V\), where \(V = \int_M v_g\). In general, \(\int_M h^X v_g\) mod \(\mathbb{Z}V\) is different from \(0\). As a matter of fact, \(\int_M h^X v_g\) appears as the leading term in the asymptotic expansion of the trace of the induced linear action of \(S^1\) in the spaces of holomorphic sections \(H^0(M, L^k)\), cf. Proposition 8.8.5 below.

The trace, \(w_X(L)\), of the restriction of \(\mathcal{L}_X\) — or rather of the corresponding hermitian operator \(-i\mathcal{L}_X\) — to \(E = H^0(M, L)\) is defined by

\[
w_X(L) = -i \sum_{j=0}^{N} \int_M (\mathcal{L}_X s_j, s_j) v_g, 
\]

for any orthonormal basis \(s = \{s_0, \ldots, s_N\}\) of \(E\). Equivalently, \(w_X(L)\) is the weight of the induced action of \(S^1\) on the complex line \(\Lambda^{N+1}(E^*)\).
Proposition 8.8.5. The trace, \( w_X(L) \), of the restriction of \(-i\mathcal{L}_X \) to \( H^0(M, L) \) is given by

\[
w_X(L) = \int_M (h_X + \frac{1}{2} \Delta h_X) \sum_{j=0}^N |s_j|^2 v_g,
\]

for any orthonormal basis \( s = \{s_0, s_1, \ldots, s_N\} \) of \( H^0(M, L) \).

Proof. From (8.8.10) and (8.8.6) we infer

\[
w_X(L) = -i \int_M \sum_{j=0}^N (\nabla_X s_j, s_j) v_g + \int_M h_X \sum_{j=0}^N |s_j|^2 v_g.
\]

For any holomorphic section \( s \) of \( L \), the hermitian inner product \( (\nabla_X s, s) \) can be written as

\[
(\nabla_X s, s) = \frac{1}{2} (d|s|^2)(X) - \frac{i}{2} (d|s|^2)(JX).
\]

Now, \( X \) is Killing, hence divergence-free, whereas \( \delta(JX) = -\Delta h_X \); we then readily infer (8.8.11). \( \square \)

We now consider the case of two hamiltonian actions of \( S^1 \) on the polarized Kähler manifold \( (M, L) \). For each generator, \( X \) and \( Y \), we choose a \( L \)-momentum, \( h_X \) and \( h_Y \). The induced action on \( H^0(M, L) \) is then given by (8.8.6). We then have:

Proposition 8.8.6. The trace, \( w_{X,Y}(L) \), of the restriction of the product \(-\mathcal{L}_X \circ \mathcal{L}_Y = \mathcal{L}_Y \circ \mathcal{L}_X \) to \( H^0(M, L) \) is given by

\[
w_{X,Y}(L) = \int_M h_X h_Y \sum_{j=0}^N |s_j|^2 v_g + \frac{1}{2} \int_M (-\Delta(h_X h_Y) + g(X, Y)) \sum_{j=0}^N |s_j|^2 v_g + \frac{1}{4} \int_M \delta((\Delta h_X) d h_Y) \sum_{j=0}^N |s_j|^2 v_g
\]

for any orthonormal basis \( s = \{s_0, \ldots, s_N\} \) of \( H^0(M, L) \).

Proof. From (8.8.6) we readily infer

\[
-(\mathcal{L}_X \circ \mathcal{L}_Y)(s) = -\nabla_X(\nabla_Y s) - idh_Y(X)s - ih_X \nabla_Y s - ih_Y \nabla_X s + h_X h_Y s,
\]

for any \( s \) in \( H^0(M, L) \). In this expression, \( dh_Y(X) = \{f_X, h_Y\} \) — cf. Section 1.2 — and \(-\langle \nabla_X(\nabla_Y s), s \rangle = -\langle X \cdot (\nabla_Y s), s \rangle + \langle \nabla_Y s, \nabla_X s \rangle = \langle \nabla_Y s, \nabla_X s \rangle \), as \( X \) is divergence-free. Since the trace of \(-\mathcal{L}_X \circ \mathcal{L}_Y \) is symmetric in \( X,Y \),
we infer

\[ w_{X,Y}(L) = \frac{1}{2} \sum_{j=0}^{N} \langle \nabla_X s_j, \nabla_Y s_j \rangle \]

(8.8.15)

\[-i \sum_{j=0}^{N} \langle h^X \nabla_Y s_j + h^Y \nabla_X s_j, s_j \rangle \]

\[+ \int_M h^X h^Y \sum_{j=0}^{N} |s_j|^2 v_g.\]

From (8.8.12), we infer that the contribution of the second line in the rhs of (8.8.15) is equal to \(-\frac{1}{2} \Delta (h^X h^Y)\); this is because, \(\delta (h^X Y + h^Y X) = -(dh^X, d^c h^Y) - (dh^Y, d^c h^X) = 0\), whereas \(\delta (h^X JY + h^Y JX) = -h^X \Delta h^Y + (dh^X, dh^Y) - h^Y \Delta h^X + (dh^X, dh^X) = -\Delta (h^X h^Y)\). In order to calculate the contribution of the first line, we first establish the following identity

\[(\nabla_X s, \nabla_Y s) + (\nabla_Y s, \nabla_X s) = \frac{1}{2}(X \cdot Y \cdot |s|^2 + JX \cdot JY \cdot |s|^2) \]

(8.8.16)

\[-g(X, Y) |s|^2,\]

for any \(s\) in \(H^0(M, L)\) and any \(X, Y\) in \(\mathfrak{h}\) (note that the rhs is symmetric in \(X, Y\), as \([JX, JY] = -[X, Y]\)); indeed, we have that

\[X \cdot Y \cdot |s|^2 = (\nabla_X (\nabla_Y s), s) + (s, \nabla_X (\nabla_Y s))\]

(8.8.17)

\[+ (\nabla_Y s, \nabla_X s) + (\nabla_X s, \nabla_Y s),\]

whereas

\[JX \cdot JY \cdot |s|^2 = (\nabla_{JX} (\nabla_{JY} s), s) + (s, \nabla_{JX} (\nabla_{JY} s))\]

\[+ (\nabla_{JY} s, \nabla_{JX} s) + (\nabla_{JX} s, \nabla_{JY} s)\]

\[= i(\nabla_{JX} (\nabla_{JY} s), s) - i(s, \nabla_{JX} (\nabla_{JY} s))\]

\[+ (\nabla_Y s, \nabla_X s) + (\nabla_X s, \nabla_Y s),\]

as \(\nabla_{JY} s = i\nabla_Y s\); moreover,

\[\nabla_{JX} (\nabla_{JY} s) = \nabla_Y (\nabla_{JX} s) + R^\nabla_{Y,JX} s + \nabla_{[JX,Y]} s\]

\[= i\nabla_Y (\nabla_X s) + ig(X, Y) s + i\nabla_{[X,Y]} s\]

\[= i\nabla_X (\nabla_Y s) - \omega(X, Y) s + ig(X, Y) s;\]

it follows that

\[JX \cdot JY \cdot |s|^2 = -(\nabla_X (\nabla_Y s), s) - (s, \nabla_X (\nabla_Y s))\]

(8.8.18)

\[+ (\nabla_Y s, \nabla_X s) + (\nabla_X s, \nabla_Y s)\]

\[- 2g(X, Y) |s|^2;\]

(8.8.16) readily from (8.8.17)-(8.8.18). Since \(X\) is divergence-free, the lhs of (8.8.17) integrates to zero, whereas

\[\int_M (JX \cdot JY \cdot |s|^2) v_g = \int_M \delta((\Delta h^X) dh^Y)|s|^2 v_g;\]

this completes the proof of (8.8.13). \(\square\)
Remark 8.8.4. The symmetry in $X, Y$ of $w_{X,Y}(L)$ is not directly apparent in the last term of the rhs of (8.8.13). This, in fact, can be rewritten in the following symmetric form:

$$
(8.8.19) \quad \frac{1}{4} \int_M (\Delta h^X \Delta h^Y - (dd^c h^X, dd^c h^Y) - \Delta(g(X,Y))) \sum_{j=0}^N |s_j|^2 v_g,
$$

as follows from the following simple computation:

$$
(8.8.20) \quad (dd^c h^X, dd^c h^Y) = (d^c h^X, \delta dd^c h^Y + \delta((\iota_X dd^c h^Y)
$$

(note that $\mathcal{L}_X(d^c h^Y) = d^c(\mathcal{L}_X h^Y)$, as $X$ is (real) holomorphic, so that $\delta(\mathcal{L}_X(d^c h^Y)) = \delta d^c(\mathcal{L}_X h^Y) = 0$).

### 8.9. The Futaki character and the Futaki-Mabuchi bilinear forms

as asymptotic invariants

In this section, we still consider a compact Kähler manifold $(M, g, \omega)$ polarized by $(L, h)$, which admits a hamiltonian $S^1$-action of generator $X$. If $h^X$ is a $L$-momentum of $X$ with respect to $\omega$ — cf. Proposition 8.8.4 and Remark 8.8.3 — then $kh^X$ is a $L^k$-momentum with respect to $k \omega$. This is because the lifted action of $S^1$ on $L$ descends to an action on $L^k$ via the map $u \mapsto \otimes^k u$ from $L$ to $L^k$ which realizes $L$ as a $k$-fold branched covering of $L^k$ and the $L^k$-momentum of $X$ corresponding to this $S^1$-action is clearly $kh^X$. For any positive integer $k$, we denote by $w_X(L^k)$ the trace — as defined above — of the induced linear action on $E_k = H^0(M, L^k)$. Note that $w_X(L^k)$ depends on the chosen $L$-momentum $h^X$; by Proposition 8.8.4, if $h^X$ is replaced by $h^X + \ell$ for some integer $\ell$, then $w_X(k)$ is replaced by $w_X(k) + \ell(N_k + 1)$. When $k$ tends to infinity, we have the following asymptotic expansion:

**Proposition 8.9.1.** Let $M$ be any compact Kähler manifold $(M, g, J, \omega)$ of complex dimension $n$, polarized by a hermitian holomorphic line bundle $(L, h)$. Consider any hamiltonian isometric action of $S^1$ on $M$ and $h^X$ an $L$-momentum for the generator $X$. Then the trace $w_X(L^k)$ of the corresponding linear action on $H^0(M, L^k)$ admits an asymptotic expansion

$$
(8.9.1) \quad w_X(L^k) \simeq \frac{1}{(2\pi)^m} \sum_{r=0}^{+\infty} b_r^X k^{m+1-r},
$$

meaning that, for any non-negative integer $q$, we have

$$
(8.9.2) \quad w_X(L^k) = \frac{1}{(2\pi)^m} \sum_{r=0}^q b_r^X k^{m+1-r} + O(k^{m-r}).
$$

Moreover, the first two coefficients $b_0^X, b_1^X$ are given by

$$
(8.9.3) \quad b_0^X = \int_M h^X v_g, \quad b_1^X = \frac{1}{4} \int_M s_g h^X v_g.
$$
Proof. From (8.8.11), we infer that

\[ w_X(L^k) = \int_M (kh^X + \frac{1}{2} \Delta h^X) \sum_{j=0}^{N_k} |s_j^{(k)}|_{h^{(k)}}^2 k^m v_g, \]  

for any orthonormal basis \( s^{(k)} = \{ s_0^{(k)}, s_1^{(k)}, \ldots, s_{N_k}^{(k)} \} \) with respect to the hermitian inner product \( \langle \cdot, \cdot \rangle_{h^{(k)}} \) on \( H^0(M, L^k) \) determined by \( \omega_k = k \omega \) and the induced hermitian inner product \( h^{(k)} \) on \( L^k \) (notice that if \( \omega \) is replaced by \( \omega_k = k \omega \), then \( h^X \) is replaced by \( k h^X \), and \( v_g \) by \( k^m v_g \), whereas \( \Delta h^X \) remains unchanged). Proposition 8.9.1 is then a direct consequence of (8.9.4) and of Theorem 8.2.1. \( \square \)

Considering two hamiltonian \( S^1 \) actions, we similarly get:

**Proposition 8.9.2.** Let \( M \) be any compact \( \text{Kähler manifold} \) \( (M, g, J, \omega) \) of complex dimension \( m \), polarized by a hermitian holomorphic line bundle \( (L, h) \). Consider any two hamiltonian isometric actions of \( S^1 \) on \( M \) and let \( h^X, h^Y \) be \( L \)-momenta for the generators \( X, Y \) of these actions. Then the trace \( w_{X,Y}(k) \) of the corresponding linear action on \( H^0(M, L^k) \) admits an asymptotic expansion

\[ w_{X,Y}(L^k) \simeq \frac{1}{(2\pi)^m} \sum_{r=0}^{\infty} b_{r}^{X,Y} k^{m+2-r}, \]

meaning that, for any non-negative integer \( q \), we have

\[ w_{X,Y}(L^k) = \frac{1}{(2\pi)^m} \sum_{r=0}^{q} b_{r}^{X,Y} k^{m+2-r} + O(k^{m-r}). \]

Moreover, the first two coefficients \( b_0^{X,Y}, b_1^{X,Y} \) are given by

\[ b_0^{X,Y} = \int_M h^X h^Y v_g, \quad b_1^{X} = \frac{1}{4} \int_M s_g h^X h^Y v_g + \frac{1}{2} \int_M g(X, Y) v_g. \]

Proof. From (8.8.13), we infer that

\[ w_{X,Y}(L^k) = k^{m+2} \int_M h^X h^Y \sum_{j=1}^{N} |s_j^{(k)}|_{h^{(k)}}^2 v_g \]

\[ + \frac{k^{m+1}}{2} \int_M (-\Delta h^X h^Y) + g(X, Y) \sum_{j=1}^{N} |s_j^{(k)}|_{h^{(k)}}^2 v_g \]

\[ + \frac{k^m}{4} \int_M \delta((\Delta h^X) d_Y) \sum_{j=1}^{N} |s_j^{(h)}|_{h^{(k)}}^2 v_g \]

Proposition 8.9.2 is then a direct consequence of (8.9.8) and of Theorem 8.2.1. \( \square \)

**Remark 8.9.1.** Proposition 8.9.1 can also be obtained by using the \( S^1 \)-equivariant Riemann–Roch formula relative to the given \( S^1 \)-action on \( L \). In the current situation, the \( S^1 \)-equivariant Riemann–Roch reads

\[ \sum_{r=0}^{m} (-1)^r \text{tr} \left( e^{-itE_X} |_{H^r(M, L^k)} \right) = \int_M c_h S^1(L^k) Td_{S^1}(M, J), \]
for any positive integer \(k\), see [16], [26], [31]. In the rhs, \(c_{h^k}\) and \(Td_{h^k}\) denote the \(S^1\)-equivariant Chern character of \(L\) and the \(S^1\)-equivariant Todd class of \((TM, J)\) respectively or, better their representatives in the de Rham picture of the \(S^1\)-equivariant cohomology: we then have \(c_{h^k}(L^k) = e^{\frac{1}{2}(\partial_t + h^k t)}\), where \(\frac{1}{2}\partial_t + h^k t\) represents the \(S^1\)-equivariant Chern class of \(L\), relative to the chosen lifted action determined by the choice of the \(L^k\)-momentum \(f^X\), and \(t\) must be regarded as a formal variable; similarly, \(Td_{h^k}(M, J)\) is represented by \(1 + \frac{1}{2}(\frac{\rho}{2\pi} + \frac{1}{2}\Delta h^k t) + \ldots\), where \(\frac{\rho}{2\pi} + \frac{1}{2}\Delta h^k t\) represents the first \(S^1\)-equivariant Chern class of the (complex) tangent bundle \((TM, J)\) or, better, of the anti-canonical line bundle \(K^{-1}_M\), relative to the canonical lifted action (cf. Proposition 8.8.3). In this formula, both sides have to be regarded as elements of the ring \(\mathbb{R}[[t]]\) of formal power series in \(t\) with real coefficients and the formula then asserts the equality of the coefficients of \(t^\ell\) for all \(\ell\). In particular, if \(k \geq k_0\), the constant term in the lhs is equal to \(d(L^k)\), whereas the coefficient of \(t\) is equal to the weight \(w_X(L^k)\) as defined above (for \(k\) large enough).

If we are only interested in the first two dominant terms in \(k\) — namely the terms in \(k^{m+1}\) and \(k^m\) — and the first two terms in \(t\), i.e. the constant term — already computed in Remark 8.2.2 — and the term in \(t\), and if \(k\) is large enough, Formula (8.9.9) can be written as

\[
RRequivshort\tag{8.9.10}
d(L^k) + w_X(L^k) t + \ldots = \int_M \left(\frac{\frac{\rho}{2\pi} + h^k t}{m!} k^m + \frac{\frac{\rho}{2\pi} + h^k t}{(m+1)!} k^{m+1} + \ldots\right) \left(1 + \frac{\rho}{4\pi} + \frac{1}{4}\Delta h^k t + \ldots\right),
\]

where, in the rhs, we only retained the terms which contribute non trivially. We then easily recover (8.2.9) and (8.9.3), cf. [77].

By bringing together Propositions 8.9.1 and 8.2.1 we get

\[
futaki-donaldson\tag{8.9.11}
\frac{w_X(L^k)}{kd(L^k)} = h^kX + \frac{1}{4}\bar{\mathcal{F}}_\omega(-JX)k^{-1} + O(k^{-2})
\]

when \(k\) tends to \(+\infty\), where \(h^kX = \frac{\int_M h^k v_\omega}{\int_M v_\omega}\) denotes the mean value of the chosen \(L\)-momentum \(h^kX\) and \(\bar{\mathcal{F}}_\omega\) the normalized Futaki character — see Section 4.12 — of the Kähler class \([\omega] = 2\pi c_1(L)\).

\textbf{PROOF.} From Propositions 8.9.1 and 8.2.1 we readily infer

\[
w_X(L^k)
\]

\[
(8.9.12)
\frac{w_X(L^k)}{kd(L^k)} = \frac{b^X_0}{a_0} + \frac{a_0 b^X_1 - a_1 b^X_0}{a^2_0} k^{-1} + O(k^{-1}),
\]
when $k$ tends to $+\infty$. From (8.2.9) we get 

$$\frac{b_0^X}{a_0} = \frac{\int_M h^X v_g}{\int_M v_g} = \overline{h^X},$$

where $\frac{a_0 b_0^X - a_1 b_0}{a_0^2} = \frac{1}{4} \frac{1}{\int_M v_g} \int_M s_g(h^X - \overline{h^X}) v_g$; since $X = \text{grad}_w h^X$, $(f^X - \overline{h^X})$ is the real potential of the (real) vector field $-JX$; from (4.12.1), we then get 

$$\int_M s_g(h^X - \overline{h^X}) v_g = F_{[\omega]}(-JX).$$

Similarly, by bringing together Propositions 8.9.2 and 8.2.1 we get

**Proposition 8.9.4.** Let $M$ be any compact Kähler manifold $(M, g, J, \omega)$ of complex dimension $m$, polarized by a hermitian holomorphic line bundle $(L, h)$. Consider any two hamiltonian isometric actions of $S^1$ on $M$ and $L$-momenta $h^X, h^Y$ for the generators $X, Y$ of these actions. Then, 

$$\frac{1}{k^2d(L)} \left( w_{X,Y}(L^k) - \frac{w_X(L^k) w_Y(L^k)}{d(L^k)} \right)$$

admits the following expansion

$$= \frac{k^{-1}}{4V_0} \left( \int_M s_g(h^X - \overline{h^X})(h^Y - \overline{h^Y}) v_g \right)
+ 2 \int_M g(X, Y) v_g - S_1 \Omega_{-JX, -JY}) + O(k^{-2}),$$

where $B_\Omega(-JX, -JY) = \int_M (h^X - \overline{h^X})(h^Y - \overline{h^Y}) v_g$ denotes the Futaki-Mabuchi inner product of $-JX = \text{grad}_g h^X$ and $-JY = \text{grad}_g h^Y$, cf. Section 4.11.

**Proof.** From Propositions 8.9.2, 8.9.1 and 8.2.1 we infer

$$\frac{w_{X,Y}(L^k)}{k^2 d(L^k)} - \frac{w_X(L^k) w_Y(L^k)}{k^2(d(L^k))^2} = \frac{a_0 b_0^{XY} - b_0^X b_0^Y}{a_0^2}
+ k^{-1} \left( \frac{b_0^{XY}}{a_0} - \frac{b_0^{XY} a_1 + b_0^X b_0^Y + b_0^Y b_0^X}{a_0^2} + 2 \frac{b_0^X b_0^Y a_1}{a_0^3} \right)
+ O(k^{-2}).$$

By replacing the coefficients by their values given by (8.2.9)-(8.9.3)-(8.9.7), we get (8.9.13).

**Remark 8.9.2.** The fact that for any $S^1$-action on a compact polarized Kähler manifold $(M, g, J, \omega)$, generated by a hamiltonian Killing vector field $X$, the (normalized) Futaki number $F_{[\omega]}(-JX)$ appears as a coefficient in the asymptotic expansion of $\frac{w_X(k)}{kd_L(L)}$ as shown in (8.9.11) has first been observed by S. Donaldson in [77] and proved to be a fact of paramount importance in the definition of the $K$-stability introduced by G. Tian. This is because the expression $\frac{w_X(k)}{kd_L(L)}$ and its asymptotic expansion still make sense not only for smooth projective manifolds but also for a large class of singular varieties, in particular for singular fibres appearing in degenerations of $M$ used in the definition of $K$-stability.
CHAPTER 9

The scalar curvature as a momentum map

9.1. The symplectic viewpoint: the space $J_{\omega_0}$

In this chapter, $M$ will denote a (connected, oriented) compact manifold of even dimension $n = 2m$ and we choose a Kähler structure $(g_0, J_0, \omega_0)$ on $M$. The de Rham class of $\omega_0$ is denoted by $\Omega$.

In previous chapters, we fixed the complex structure $J_0$ and a Kähler class $\Omega$ on $M$ and we considered the space $M_\Omega$ of Kähler metrics $(g, J_0, \omega_0)$ on the complex manifold $(M, J_0)$ such that the de Rham class of the Kähler class $\omega$ is $\Omega$. This space is naturally acted on on the right by $H(M, J_0)$, cf. Section 3.2.

In this chapter, we instead consider the space of Kähler structures $(g, J, \omega_0)$ on $M$ whose Kähler form is fixed — equal to $\omega_0$ — and whose complex structure, $J$, belongs to the same Teichmüller class as $J_0$, meaning that there exists a diffeomorphism $\Phi$ in $\text{Diff}_0^0 M$, the identity component of the group of diffeomorphisms of $M$, such that $J = \Phi \cdot J_0$ (such a $\Phi$ is then uniquely defined up to right composition with an element of $H(M, J_0)$).

We then define $J_{\omega_0}$ as the connected component of $(g_0, J_0, \omega_0)$ in this space.

In the above, $\Phi \cdot J_0$ denotes the natural (left) action of $\Phi$ on $J_0$, so that $J(X) = (\Phi \cdot J_0)(X) = \Phi_* (J_0(\Phi^{-1}_*(X)))$, for any point $x$ of $M$ and any vector $X$ in $T_x M$. More generally, $\Phi$ will denote the left action of $\Phi$ on any kind of tensor fields; in particular, the right action $(\omega, \gamma) \mapsto \gamma^* \omega$ of $H(M, J_0)$ on $\mathcal{M}_{J_0}$ can also be written as $(\omega, \gamma) \mapsto \gamma^{-1} \cdot \omega$.

The identity component, $\text{Sp}_0(M, \omega_0)$, of the group of symplectomorphisms of $(M, \omega_0)$ acts naturally — on the left — on $J_{\omega_0}$, by $(\psi, J) \mapsto \psi \cdot J$.

In contrast with $H(M, J_0)$, which is a (finite-dimensional) complex Lie group, $\text{Sp}_0(M, \omega_0)$ is an infinite dimensional “Lie group”, whose Lie algebra, denoted by $\mathfrak{sp}$, is the space of real vector fields $X$ such that $L_X \omega_0 = 0$.

Equivalently, $\mathfrak{sp}$ is the space of real vector fields $X$ whose symplectic dual 1-form $X^{\flat \omega_0} := -\iota_X \omega_0$ is closed (cf. Cartan formula (1.4.4)).

We denote by $\text{ham}$ the Lie subalgebra of hamiltonian vector fields, defined as the space of elements $Z$ of $\mathfrak{sp}$ such that $Z^{\flat \omega_0} = df^Z$, for some real function $f^Z$ called the momentum of $Z$ with respect to $\omega_0$; alternatively, $Z = \text{grad}_{\omega_0} f^Z$, where $\text{grad}_{\omega_0}$ denotes the symplectic gradient with respect to $\omega_0$, cf. Section 1.2. Unless otherwise specified, the momentum $f^Z$ will be normalized by $\int_M f^Z \omega_0^n = 0$. The corresponding subgroup of $\text{Sp}_0(M, \omega_0)$ will be denoted by $\text{Ham}(M, \omega_0)$.

$1$ Roughly speaking, $\text{Ham}(M, \omega_0)$ is defined as the set of those elements of $\text{Sp}_0(M, \omega_0)$ which are the time one value of the flow — in the sense of (4.6.4) — of time dependent
The spaces $\mathcal{M}_\Omega$ and $\mathcal{J}_{\omega_0}$ are closely related according to the following proposition:

**Proposition 9.1.1.** (i) There exists an $\text{Sp}_0(M, \omega_0)$-invariant map, $p$, from $\mathcal{J}_{\omega_0}$ to the quotient $\mathcal{M}_\Omega/\text{H}(M, J_0)$ and an $\text{H}(M, J_0)$-invariant map, $q$, from $\mathcal{M}_\Omega$ to the quotient $\mathcal{J}_{\omega_0}/\text{Sp}_0(M, \omega_0)$, which are inverse to each other, hence induce a natural identification

$$
\mathcal{J}_{\omega_0}/\text{Sp}_0(M, \omega_0) \xrightarrow{p} \mathcal{M}_\Omega/\text{H}(M, J_0).
$$

The maps $p$ and $q$ are defined as follows. For any $J$ in $\mathcal{J}_{\omega_0}$ there exists $\Phi$ in $\text{Diff}_0(M)$, uniquely defined up to left composition by an element of $\text{H}(M, J_0)$, such that $J = \Phi \cdot J_0$ and we then set

$$
p(J) = \Phi^* \omega_0 = \Phi^{-1} \cdot \omega_0 \mod \text{H}(M, J_0).
$$

Similarly, if $\omega$ belongs to $\mathcal{M}_\Omega$, by Moser lemma — cf. Section 4.6 — there exists $\Phi$ in $\text{Diff}_0(M)$, uniquely defined up to right composition by an element of $\text{Sp}_0(M, \omega_0)$, such that $\omega = \Phi^* \omega_0$ and we then set

$$
q(\omega) = \Phi \cdot J_0 \mod \text{Sp}_0(M, \omega_0).
$$

(ii) For any $J$ in $\mathcal{J}_{\omega_0}$ the tangent space $T_J \mathcal{J}_{\omega_0}$ is described by

$$
T_J \mathcal{J}_{\omega_0} = \{ A = -\mathcal{L}_Z J \mid Z \in \mathfrak{sp} + J \mathfrak{sp} = \mathfrak{sp} \oplus \mathfrak{J \text{ham}} \}.
$$

Via this identification, the differential of $p$ at $J = \Phi \cdot J_0$ is given by

$$
dp(-\mathcal{L}_Z J) = \Phi^*(\mathcal{L}_Z \omega_0) = dd^c(\Phi^* f)
$$

for each $Z = X - JY$ in $\mathfrak{sp} \oplus J \mathfrak{\text{ham}}$, where $f$ denotes the normalized momentum of the hamiltonian vector field $Y$, hence is determined by

$$
\mathcal{L}_Z \omega_0 = dd^c f
$$

and $\int_M f \omega_0^n = 0$ (here $dd^c$ stands for the operator $J_0 d J_0^{-1}$), also denoted by $d^c$, whereas $d^c := J_0 d J_0^{-1}$.

(iii) $\mathcal{J}_{\omega_0}$ is a complex, hence Kähler, submanifold of $\mathfrak{C}_{\omega_0}$.

**Proof.** (i) The symplectic form $\omega$ defined by (9.1.2) is $J_0$-invariant, as the pair $(J_0, \omega)$ is the image of the Kähler pair $(J, \omega_0)$ by $\Phi^{-1}$, and belongs to $\Omega$ as $\Phi$ sits in $\text{Diff}_0 M$. Similarly, the complex structure $J$ defined by (9.1.3) is compatible with $\omega_0$, hence belongs to $\mathcal{J}_{\omega_0}$ as $\Phi$ sits in $\text{Diff}_0 M$ and the pair $(J, \omega_0)$ is the image of the Kähler pair $(J_0, \omega)$ by $\Phi$. The maps $p$ and $q$ are then well defined, from $\mathcal{J}_{\omega_0}$ to $\mathcal{M}_\Omega/\text{H}(M, J_0)$ and from $\mathcal{M}_\Omega$ to $\mathcal{J}_{\omega_0}/\text{Sp}_0(M, \omega_0)$ respectively. Moreover, $p$ and $q$ are evidently $\text{Sp}_0(M, \omega_0)$- and $\text{H}(M, J_0)$-invariant respectively. They then determine maps from $\mathcal{J}_{\omega_0}/\text{Sp}_0(M, \omega_0)$ to $\mathcal{M}_\Omega/\text{H}(M, J_0)$ and from $\mathcal{M}_\Omega/\text{H}(M, J_0)$ to $\mathcal{J}_{\omega_0}/\text{Sp}_0(M, \omega_0)$, which are clearly inverse to each other (The Moser lemma used to define $q$ has been described in Section 4.6). Note that for any $J = \Phi \cdot J_0$ in $\mathcal{J}_{\omega_0}$ the riemannian metric $g := \omega_0(\cdot, J \cdot)$ determined by the Kähler pair $(J, \omega_0)$ and the riemannian metric $\tilde{g} := \omega(\cdot, J_0 \cdot)$ determined vector fields of the form $X_t = \text{grad}_\omega H_t$, defined on $M \times [0, 1]$, see, e.g., [164] for a detailed exposition.
by the Kähler pair \((J_0, \omega)\), where \(\omega\) is defined by (9.1.2), are linked together by

\[ g = \Phi^* g = \Phi^{-1} \cdot g. \]

(ii) Let \(J_t = \Phi_t \circ J_0\) be a curve in \(\mathcal{J}_{\omega_0}\), where \(\Phi_t\) is a curve in \(\text{Diff}_0 M\), with \(\Phi_0 = \Phi\). We then have

\[ j := \frac{d}{dt}_{|t=0} \Phi_t \circ J_0 = -\mathcal{L}_Z J, \]

with \(Z := \frac{d}{dt}_{|t=0} (\Phi_t \circ \Phi^{-1})\). Moreover, the fact that \(J_t\) belongs to \(\mathcal{J}_{\omega_0}\) for all \(t\) implies that \(\mathcal{L}_Z J\) is symmetric with respect to the metric \(g\) determined by \((J, \omega_0)\). If \(\zeta\) denotes the riemannian dual 1-form of \(Z\) with respect to the metric \(g\), this condition just means that \(d\zeta\) is \(J\)-invariant (cf. Lemma 1.23.2). By Lemma 1.17.1, the latter condition is equivalent to the fact that the Hodge decomposition of \(\zeta\) with respect to \(g\) is of the form

\[ \zeta = \zeta_H + df + d^J h, \]

where \(\zeta_H\) is \(g\)-harmonic — equivalently, \(\zeta_H\) is the real part of a \(J\)-holomorphic 1-form — and \(d^J h := Jdh\). Notice that the symplectic dual 1-form of \(Z\) with respect to \(\omega_0\) is equal to \(J\zeta = J\zeta_H + df - d\zeta\). It follows that \(\zeta\) is of the form (9.1.9) if and only if \(Z = X - JY\) where \(X, Y\) both belong to \(\mathfrak{sp}\): if so, \(X, Y\) are not uniquely determined by \(Z\), because of the presence of the harmonic component \(\zeta_H\) in (9.1.9); they can however be made uniquely determined by demanding that \(Y\) be hamiltonian; then, \(f\) is the normalized momentum of \(Y\), also given by (9.1.6). Conversely, \(Z\) satisfies (9.1.6), for some real function \(f\), if and only if \(J\zeta - df\) is closed, if and only if \(\zeta\) is of the form (9.1.9). It follows that the tangent space \(T_J \mathcal{J}_{\omega_0}\) is described by (9.1.4). For any such \(Z\), we then have:

\[
d p(-L_Z J) = \frac{d}{dt}_{|t=0} \Phi^* (\Phi_t \circ \Phi^{-1})^* \omega_0 \\
= \Phi^* (L_Z \omega_0) = \Phi^* (df) \\
= dd\Lambda (\Phi^* f) \cong \Phi^* f.
\]

This gives (9.1.5). Note that the chosen normalization \(\int_M f \omega_0^m = 0\) for the momentum \(f\) implies the usual normalisation \(\int_M \Phi^* f \omega^m = 0\) for the real function \(\Phi^* f\) viewed as an element of the tangent space \(T_\omega \mathcal{M}_\Omega\). Since the latter space is made of \(all\) real functions normalized that way we then derive from (9.1.5) that \(T_J \mathcal{J}_{\omega_0} = \{ A = -L_Z J \} \) for \(all\) vector fields \(Z\) in \(\mathfrak{sp} + J\mathfrak{sp}\).

As a subspace of \(T_J \mathcal{J}_{\omega_0}\), the tangent space of the orbit \(O\) of \(J\) under the action of \(\text{Sp}(M, \omega_0)\) is the space of \(L_X J\) for any \(X\) in \(\mathfrak{sp}\). From Proposition 9.1.1 we readily infer

\[ T_J \mathcal{J}_{\omega_0} = T_J O + \mathbb{J}(T_J O). \]

This sum is not direct in general. More precisely:

**Proposition 9.1.2.** For any \(J\) in \(\mathcal{J}_{\omega_0}\), the intersection \(T_J O \cap \mathbb{J}(T_J O)\) is the space of elements of the form \(L_Z J\), for all elements \(Z\) of \(\mathfrak{sp}\) of the form \(Z = Z_H + \text{grad}_{\omega_0} f^X\), where \(Z_H\) is the dual vector field of a harmonic
1-form and \( f^X_g \) the real potential of a (real) holomorphic vector field with respect to \( J \).

In particular, the sum (9.1.10) is direct if and only if any \( J \)-harmonic 1-form is \( g \)-parallel, where \( g \) denotes the associated riemannian metric \( g = \omega_0 \cdot (\cdot, J \cdot) \), and if \( h(M, J) \) splits as in (3.6.1).

**Proof.** Elements of \( T_j \mathcal{O} \) are of the form \( \mathcal{L}_Z J \) for \( Z_H + \text{grad}_{\omega_0} h Z^J \) in \( \mathfrak{sp} \), where \( Z_H \) is the dual vector field of a harmonic 1-form. Such an element belongs to \( \mathcal{J}(T_j \mathcal{O}) \) if and only if there exists \( Z_1 = Z_{1, H} + \text{grad}_{\omega_0} h Z^J \) in \( \mathfrak{sp} \) such that \( \mathcal{L}_Z J = \mathcal{J} \mathcal{L}_Z J = \mathcal{L}_{JZ_1} J \). Equivalently, \( Z - JZ_1 = (Z_H - JZ_{1, H}) + \text{grad}_{\omega_0}(h Z^J) + \text{grad}_L Z^J \) is a \( (\text{real}) \) holomorphic vector field. In particular, \( h Z^J \) is the real potential of the \( (\text{real}) \) holomorphic vector field \(- (Z_1 + JZ)\). Conversely, if \( Z = Z_H + \text{grad}_{\omega_0} h Z^J \), where \( Z_H \) is the dual vector field of a harmonic 1-form and \( h Z^J \) is the real potential of a \( (\text{real}) \) vector field, \( X = X_H + \text{grad}_L Z^J + \text{grad}_{\omega_0} h X \), say, then \( \mathcal{L}_{JZ_1} J = \mathcal{J}_{JZ_1} J \), with \( Z_1 = -(X_H + JZ_H + \text{grad}_{\omega_0} h X) \). The last statement follows readily. \( \square \)

**Remark 9.1.1.** In view of (9.1.10) and of Proposition 9.1.1, \( J_{\omega_0} \) appears as a complexification of the orbit of \( J_0 \) under the action of the symplectomorphism group \( \text{Sp}_0(M, \omega_0) \) or as a “complex orbit” of an \( (\text{inexisting}) \) complexification of \( \text{Sp}_0(M, \omega_0) \) in the space of all almost complex structures compatible with \( \omega_0 \) considered in the next section, cf. [73], [74]. The quotient \( J_{\omega_0}/\text{Sp}_0(M, \omega_0) \) — hence also the quotient \( M_{\Omega}/H(M, J_0) \) by Proposition 9.1.1 — can then be regarded as dual symmetric space of the group \( \text{Sp}_0(M, \omega_0) \), cf. [74].

**Remark 9.1.2.** For any fixed hamiltonian vector field \( Z = \text{grad}_{\omega_0} \), denote by \( \hat{Z} \) the induced vector field on \( J_{\omega_0} \), defined by \( \hat{Z}(J) = -\mathcal{L}_Z J. \) By its very definition, \( \hat{Z} \) is tangent to the orbit \( \mathcal{O} \) and has a flow, \( \Phi^Z_t \), defined by \( \Phi^Z_t(J) = \Phi^Z_t \cdot J \), where \( \Phi^Z_t \) stands for the flow of \( Z \) on \( M \). On the other hand, the existence of a flow for the vector field \( \mathcal{J} \hat{Z} \) is not granted a priori on \( J_{\omega_0} \). This is actually closely related to the existence issue of geodesics on \( M_{\Omega} \), as shown by the following proposition, communicated to me by D. Calderbank:

**Proposition 9.1.3.** Let \( \omega_t = \omega_0 + dd^c \phi_t \) be a geodesic in \( M_{\Omega} \) starting from \( \omega_0 \), with tangent vector \( \phi_0 \). Let \( \Phi_t \) be the corresponding Moser lift — cf. Section 4.6 — in \( \text{Diff}_0(M) \), so that \( \Phi_t \omega_t = \omega_0 \). Then, \( J_t := \Phi_t^{-1} \cdot J_0 \) is an integral curve in \( J_{\omega_0} \) of \( \mathcal{J} \hat{Z} \), with \( Z = \text{grad}_{\omega_0} \phi_0 \).

**Proof.** By definition, \( \Phi_t \) is the “flow”, as defined in (4.6.4), of the 1-parameter vector field \( X_t = -\text{grad}_{g_t} \phi_t \), with \( g_t := \omega_t(\cdot, J_0 \cdot) \), see Section 4.6. By (4.6.5), we have \( J_t = \mathcal{L}_{\Phi_t^{-1}} X_t J_t \), whereas the 1-parameter vector field \( \Phi_t^{-1} \cdot X_t \) is given by:

\[
\Phi_t^{-1} \cdot X_t = -\Phi_t^{-1} \cdot \text{grad}_{g_t} \phi_t \\
= -\text{grad}_{\Phi_t^{-1} \cdot g_t} (\Phi_t^{-1} \cdot \phi_t) \\
= -\text{grad}_{g_t} \phi_0 \\
= J_t \text{grad}_{\omega_0} \phi_0.
\] (9.1.11)
Here, $\tilde{g}_t := \Phi_t^{-1} \cdot g_t$ stands for the metric determined by the pair $(\omega_0, J_t)$, which is the image by $\Phi_t^{-1}$ of the pair $(\omega_t, J_0)$, and we used the fact that $\Phi_t^{-1} \cdot \dot{\phi}_t = \Phi_t^* \dot{\phi}_0 = \dot{\phi}_0$, as $\Phi_t$ is the Moser lift of the geodesic $\omega_t = \omega_0 + d\theta^\nu \phi_t$, cf. (4.6.8); in particular, $\text{grad}_{\tilde{g}_t} \phi_0 = -J_t \text{grad}_{g_0} \phi_0$. We thus get $J_t = \mathcal{L}_{J_t \text{grad}_{\omega_0} \phi_0} J_t = J \tilde{Z}(J_t)$.

\section{The space of symplectic almost complex structures as a Kähler manifold}

In this section, we fix a symplectic form $\omega_0$ on $M$. An almost complex structure $J$ is said to be \textit{compatible with} $\omega_0$, or simply \textit{symplectic}, if $g := \omega_0(\cdot, J\cdot)$ is a riemannian metric. The space of all symplectic almost-structures is denoted by $\mathcal{AC}_{\omega_0}$. This space is then the space of all (smooth) sections of the fiber bundle over $M$ whose fiber at $x$ is the space $C(\omega_0(x))$ of all complex structures of $T_x M$ compatible with $\omega_0(x)$, which is a hermitian symmetric space of non-compact type, cf. Appendix B.2. In particular, $C(\omega_0(x))$ is contractible, so that $\mathcal{AC}_{\omega_0}$ is itself contractible. Via the Cayley transformation $J \mapsto \mu$ relative to any chosen $J_0$ — cf. Appendix B.1 — $\mathcal{AC}_{\omega_0}$ can also be viewed as an open set in the space of sections of the (complex) vector bundle of $J_0$-anti-commuting endomorphisms of $(TM, J_0)$. This will be used in the proof of Proposition 9.2.1 below.

The subspace of integrable almost complex structures in $\mathcal{AC}_{\omega_0}$ will be denoted by $\mathcal{C}_{\omega_0}$; $\mathcal{J}_{\omega_0}$ is then a subspace of $\mathcal{C}_{\omega_0}$. By definition, a riemannian metric $g$ is attached to any $J$ in $\mathcal{AC}_{\omega_0}$ in such a way that the triple $(g, J, \omega_0)$ is an almost Kähler structure. The volume form $v_g$ of $g$ coincides with the symplectic volume form $\frac{\omega^m}{m!}$ and is then independent of $J$ in $\mathcal{AC}_{\omega_0}$.

The groups $\text{Sp}_0(M, \omega_0)$ and $\text{Ham}(M, \omega_0)$ naturally act (on the left) on both spaces $\mathcal{AC}_{\omega_0}$ and $\mathcal{C}_{\omega_0}$. As in the previous section, for any vector field $Z$ in $\mathfrak{sp}$, we denote by $\tilde{Z}$ the induced vector field on $\mathcal{AC}_{\omega_0}$, defined by

\begin{equation}
\hat{Z} J = -\mathcal{L}_Z J.
\end{equation}

An alternative useful expression is given by

**Lemma 9.2.1.** For any almost Kähler structure $(g, J, \omega)$, the Lie derivative of $J$ along any vector field $Z$ preserving $\omega$ has the following expression

\begin{equation}
\mathcal{L}_Z J = -DZ + (DZ)^* J = J(DZ + (DZ)^*),
\end{equation}

where $D$ denotes the Levi-Civita connection of the metric $g$ and $(DZ)^*$ the adjoint of $DZ$ with respect to $g$ as a field of endomorphisms of $TM$.

**Proof.** Since $\mathcal{L}_Z \omega = 0$ and $\omega(X, Y) = g(JX, Y)$, we have

\begin{align*}
0 &= (\mathcal{L}_Z \omega)(X, Y) \\
&= (\mathcal{L}_Z g)(JX, Y) + g((\mathcal{L}_Z J)X, Y) \\
&= g(DJX, Y) + g(JX, D_Y Z) + g((\mathcal{L}_Z J)X, Y) \\
&= g((DZ + (DZ)^*)(JX), Y) + g((\mathcal{L}_Z J)X, Y)
\end{align*}

for any vector fields $X, Y$. The second expression in (9.2.2) follows from the fact that $\mathcal{L}_Z J$ anticommute to $J$ for any vector field $Z$. \qed
We denote by Aut$(TM, \omega_0)$ the group bundle of all automorphisms $\gamma$ of $TM$ which preserve $\omega_0$ in the sense that $\omega_0(\gamma X, \gamma Y) = \omega_0(X, Y)$ for any vector fields $X, Y$, and by End$(TM, \omega_0)$ the corresponding Lie algebra vector bundle, whose elements are characterized by $\omega_0(AX, Y) + \omega_0(X, AY) = 0$. We denote by $\mathcal{G}_{\omega_0}$, resp. $\mathcal{L}_{\omega_0}$, the space of smooth sections of Aut$(TM, \omega_0)$, resp. of End$(TM, \omega_0)$: $\mathcal{G}_{\omega_0}$ can be viewed as an (infinite dimensional) Lie group whose Lie algebra is $\mathcal{L}_{\omega_0}$. For any $a$ in $\mathcal{L}_{\omega_0}$ we denote by $\exp a$ the exponential of $a$ in $\mathcal{G}_{\omega_0}$.

The space $\mathcal{A}_{\omega_0}$ is naturally embedded in both $\mathcal{G}_{\omega_0}$ and $\mathcal{A}_{\omega_0}$. In particular, at each point $J$ of $\mathcal{A}_{\omega_0}$, the tangent space $T_J\mathcal{A}_{\omega_0}$ is naturally identified with the space of sections of End$(TM, \omega_0)$ which anticommutes with $J$, i.e. the space of field of endomorphisms of $TM$ which anticommute with $J$ and are symmetric with respect to the metric $g = \omega_0(\cdot, J \cdot)$.

The group $\mathcal{G}_{\omega_0}$ acts transitively on the space $\mathcal{A}_{\omega_0}$ by

$$\gamma \cdot J = \gamma J \gamma^{-1},$$

for any $J$ in $\mathcal{A}_{\omega_0}$ and any $A$ in $\mathcal{G}_{\omega_0}$. More precisely, we have

LEMMA 9.2.2. For any pair $J_0, J$ of elements of $\mathcal{A}_{\omega_0}$, there exists a unique $a$ in $\mathcal{L}_{\omega_0}$ which anticommutes with both $J_0$ and $J$ and joins $J_0$ to $J$ in the following sense:

$$J = \exp(a) J_0 \exp(-a).$$

PROOF. We denote by $g = \omega_0(\cdot, J \cdot)$ and $g_0 = \omega_0(\cdot, J_0 \cdot)$ the riemannian metrics attached to $J$ and $J_0$ respectively. Define $S$ by $J = J_0 S$; then, $S$ belongs to $\mathcal{G}_{\omega_0}$, as both $J, J_0$ do, and is positive definite with respect to $g$ and $g_0$, as $g_0(SX, Y) = g(X, Y)$ and $g(SX, Y) = g_0(X, Y)$. It follows that $S = \exp(2a)$ for a unique $a$ in $\mathcal{L}_{\omega_0}$ which is again symmetric with respect to $g$ and $g_0$, hence anticommutes with $J$ and $J_0$. We thus have: $J = \exp(2a) J_0 = \exp(a) J_0 \exp(-a)$. Conversely, if (9.2.4) is satisfied where $a$ anticommutes with $J_0$, then $J_0 \exp(-a) = \exp(a) J_0$ and we get $J = \exp(2a) J_0$, so that $\exp(a)$ is the positive square root of $S = -J_0 J$.

Any section, $a$, of End$(TM, \omega_0)$ determines a vector field, $\hat{a}$, on $\mathcal{A}_{\omega_0}$ defined by

$$\hat{a}(J) = \frac{d}{dt} \bigg|_{t=0} \exp(-ta) J \exp(ta) = [J, a],$$

for any $J$ in $\mathcal{A}_{\omega_0}$. Conversely, for any $J$ in $\mathcal{A}_{\omega_0}$, any $\hat{J}$ in $T_J\mathcal{A}_{\omega_0}$ is of the form $\hat{J} = \hat{a}(J)$, for some section $a$ of End$(TM, \omega_0)$. Such an $a$ is well defined up to the addition of any section of End$(TM, \omega_0)$ which commutes to $J$; in particular, $a$ is uniquely defined by demanding that $a$ anticommute to $J$. We then have

$$a = -\frac{1}{2} [J, \hat{J}].$$

Since $\hat{a}$ is pointwise induced by a right action of Aut$(TM, \omega_0)$, we have

$$[\hat{a}, \hat{b}] = [\hat{a}, b],$$

for any two sections $a, b$ of End$(TM, \omega_0)$ (in the lhs, the bracket stands for the bracket of the two vector fields $\hat{a}, \hat{b}$ on $\mathcal{A}_{\omega_0}$, whereas the bracket in the
rhs denotes the commutator \([a, b] = ab - ba\). This computation can also
easily be done directly in the vector space of sections of \(\text{End}(TM, \omega_0)\).

The actions of \(G_{\omega_0}\) and \(\text{Sp}(M, \omega_0)\) on \(AC_{\omega_0}\) do not commute but \(\text{Sp}(M, \omega_0)\)
acts on \(G_{\omega_0}\) and on \(L_{\omega_0}\) in an obvious way. In particular, for any \(Z\) in \(sp\) and any \(a\) in \(L_{\omega_0}\), we have

\[
[\hat{Z}, \hat{a}] = \hat{L}_Z a.
\]

The space \(AC_{\omega_0}\) comes naturally equipped with an almost hermitian
structure, whose almost complex structure \(J\) is defined by

\[
JA = J \circ A,
\]

the metric \(g\) by

\[
g(A, B) = \frac{1}{2} \int_M \text{tr}(AB) v_g,
\]

and the Kähler form, \(\kappa\), by

\[
\kappa_J(A, B) = \frac{1}{2} \int_M \text{tr}(JAB)v_g.
\]

The following identities are easily checked:

\[
[\hat{a}, J\hat{b}] = J[\hat{a}, \hat{b}], \quad [\hat{J}a, \hat{J}b] = -[\hat{a}, \hat{b}],
\]

for any two sections \(a, b\) of \(\text{End}(TM, \omega_0)\). Together with (9.2.7), they imply
that the vector fields \(\hat{a}\) preserve \(J\).

**Proposition 9.2.1.** The almost complex structure \(J\) is integrable.

**Proof.** For any chosen \(J_0\) in \(AC_{\omega_0}\) the Cayley transformation \(J \mapsto \mu = (J + J_0)^{-1}(J - J_0)\)
realizes \(AC_{\omega_0}\) as an open subset of the (complex) vector space of symmetric
endomorphisms of \(TM\) which anti-commute to \(J_0\), namely the open set of
those endomorphisms \(B\) which satisfy the additional (open) condition that
\(I + B\) and \(I - B\) are both invertible. Moreover, by (B.1.7), the induced
(integrable) complex structure coincides with \(J\). \(\square\)

**Proposition 9.2.2.** The Kähler form \(\kappa\) is closed.

**Proof.** The exterior differential \(d\kappa\), is convenienly computed by using
vector fields of the form \(\hat{a}\), defined by (9.2.5), so that

\[
kappa_J(\hat{a}, \hat{b}) = \int_M \text{tr}(J[a, b]) v_g,
\]

and the general formula

\[
d\kappa(\hat{a}, \hat{b}, \hat{c}) = \hat{a} \cdot \kappa(\hat{b}, \hat{c}) + \hat{b} \cdot \kappa(\hat{c}, \hat{a}) + \hat{c} \cdot \kappa(\hat{a}, \hat{a})
- \kappa([\hat{a}, \hat{b}], \hat{c}) - \kappa([\hat{b}, \hat{c}], \hat{a}) - \kappa([\hat{c}, \hat{a}], \hat{b}),
\]

for any (fixed) three sections \(a, b, c\) of \(\text{End}(TM, \omega_0)\). By using (9.2.13) and
(9.2.7), we easily check that the two lines in the rhs of (9.2.14) are separately
equal to 0. \(\square\)
Since $\mathbf{J}$ is integrable and $\kappa$ is closed, the natural almost hermitian structure of $\mathcal{A}_{\omega_0}$ can be considered as a Kähler structure. In fact, we can easily define a “Levi-Civita connection”, say $\mathcal{D}$, by first defining $\mathcal{D}\hat{a}$ some distinguished vector fields like $\hat{a}$, $\hat{Z}$ or their image by $\mathbf{J}$, by using the general Koszul formula (1.1.1), then, for any vector field $\xi$, by metric duality, e.g.

\[(9.2.15) \quad ((\mathcal{D}_{A}\xi, \hat{a})) = A \cdot ((\xi, \hat{a})) - ((\xi, \mathcal{D}_A \hat{a})),\]

for any $J$ in $\mathcal{A}_{\omega_0}$ and any vector $A = \hat{J}$ in $T_J \mathcal{A}_{\omega_0}$, by using the fact that the $\hat{a}$’s generate the tangent bundle of $\mathcal{A}_{\omega_0}$. We already used a similar procedure in Section 4.4 in order to define the Levi-Civita connection of the Mabuchi metric on $\mathcal{M}_\Omega$. We get that way, at $J$:

\[(9.2.16) \quad \mathcal{D}_A \hat{a} = [A, a^+] + \mathcal{D}_A \hat{Z} = -\mathcal{L}_Z A^-,\]

by adopting the following convention: for any $J$ in $\mathcal{A}_{\omega_0}$, any $a$ in $\mathcal{L}_{\omega_0}$ splits as $a = a^+ + a^-$, where $a^+ = \frac{1}{2}(a - J a J)$ commutes and $a^- = \frac{1}{2}(a + J a J)$ anticommutes with $J$ (note that this splitting makes sense as $JaJ$ belongs to $\mathcal{L}_{\omega_0}$ whenever $a$ does). We also get $\mathcal{D}J \hat{a} = J \mathcal{D}\hat{a}$, so that

\[(9.2.17) \quad \mathcal{D}J = 0.\]

This confirms our claim that $\mathcal{A}_{\omega_0}$ is Kähler. Of course, this Kähler structure is a “globalization” of the (symmetric) Kähler structure of the algebraic model $\mathcal{C}(\omega_0) = \text{Sp}(\omega_0)/\text{U}(J_0)$ described in Appendix B.

**Proposition 9.2.3.** The spaces $\mathcal{C}_{\omega_0}$ and $\mathcal{J}_{\omega_0}$ are Kähler subspaces of $\mathcal{A}_{\omega_0}$.

**Proof.** In both cases, it is sufficient to show that the tangent space is stable for $\mathbf{J}$. For any $J$ in $\mathcal{C}_{\omega_0}$, $T_J \mathcal{C}_{\omega_0}$ is defined as the subspace of $T_J \mathcal{A}_{\omega_0}$ defined by the condition


for any vector fields $X, Y$, obtained by linearizing the integrability condition $N(X, Y) = 0$. Equivalently, $J$ can be viewed as $(TM, J)$-valued $(0, 1)$-form and, by considering the canonical Cauchy-Riemann operator $\partial^TM$ defined in the next section, the above condition reads

\[(9.2.19) \quad d\partial^TM J = 0.\]

It is a simple exercise to check — by using $N = 0$ — that (9.2.18) is satisfied by $\hat{J}$ if and only if it is satisfied by $J\hat{J}$; it follows that $J(T_J \mathcal{C}_{\omega_0}) = T_J \mathcal{C}_{\omega_0}$. When $J$ is integrable, we have that $\mathcal{L}_{JX} J = J\mathcal{L}_X J$ for any vector field $Z$, cf. Lemma 1.1.1. This, together with (9.1.4), implies that $J(T_J \mathcal{J}_{\omega_0}) = \mathcal{J}_{\omega_0}$. 

The action of the group $\mathfrak{G}_{\omega_0}$ on $\mathcal{A}_{\omega_0}$ is clearly hamiltonian and its momentum map is simply the natural inclusion of $\mathcal{A}_{\omega_0}$ in $\mathcal{L}_{\omega_0}$, when the latter is considered as a part of the dual $\mathcal{L}_{\omega_0}^*$ via the duality induced by the metric $\mathbf{g}$.

The action of $\text{Sp}(M, \omega_0)$ on $\mathcal{A}_{\omega_0}$ is symplectic, i.e. preserves $\kappa$, but is presumably not hamiltonian. On the other hand, the action of $\text{Ham}(M, \omega_0)$ is hamiltonian and its momentum can be identified with the *hermitian scalar curvature* : this is the subject of the next sections.
Remark 9.2.1. The natural Kähler structure of \( \mathcal{A}_{C\omega_0} \) described in this section first appeared in Fujiki’s paper [84]. The fact — explained and generalized in Section 9.6 below — that the scalar curvature is a momentum for restriction of the action of \( \text{Ham}(M,\omega_0) \) on \( C\omega_0 \) was also first observed in [84].

9.3. The Chern connection of an almost Kähler structure

Proposition 9.3.1. The tangent space of any almost complex manifold \((M,J)\) admits a canonical Cauchy-Riemann operator, given by

\[
\overline{\partial}_X^{TM} Z = 2\Re([X^{0,1}, Z^{1,0}])
\]

for any two vector fields \(X, Z\). Equivalently

\[
\overline{\partial}_X^{TM} Z = \frac{1}{4}([X, Z] + [JX, JZ] + J[JX, Z] - J[X, JZ])
\]

Removable

Proposition 9.3.1.

Proof. Recall that the general notion of Cauchy-Riemann operator for any complex vector bundle \( E \) over any almost complex manifold \((M,J)\) has been introduced in Section 1.6. In the current case, \( E = (TM, J) \) is the tangent bundle itself, viewed as a complex vector bundle via \( J \). Moreover, the complex tensor product \((TM, J) \otimes \mathbb{C} \Lambda^{0,1}M\) is naturally identified with the bundle \( \text{End}^{-}(TM) \) of endomorphisms of \( TM \) which anticommute to \( J \): a Cauchy-Riemann operator can be viewed as an operator acting on (real) vector fields and taking its values in \( \text{End}^{-}(TM) \). The operator \( \overline{\partial}_X^{TM} \) defined by (9.3.1) is then a Cauchy-Riemann operator, called the canonical Cauchy-Riemann operator of \((M,J)\). The equivalence of the various expressions in (9.3.2) with the right-hand side of (9.3.1) is easily checked.

Remark 9.3.1. When \( J \) is integrable, (9.3.1) is the same as (1.8.2) and (9.3.2) then reduces to (1.8.3). In this case, a (real) vector field \( Z \) belongs to the kernel of \( \overline{\partial}_X^{TM} \) if and only if it preserves \( J \) and its \((1,0)\) part \( Z^{1,0} \) is then a holomorphic section of \( T^{1,0}M \) with respect to the natural holomorphic structure of \( T^{1,0}M \) determined by any local holomorphic coordinate system, cf. Section 1.8. When \( J \) is a general almost-complex structure, the space of “holomorphic” (real) vector fields, defined as the kernel of \( \overline{\partial}_X^{TM} \) — does no longer coincide in general with the space of (real) vector fields which preserve \( J \). As a matter of fact, the latter space is no longer preserved in general by the action of \( J \).

The complex dual of \((TM,J) \cong T^{1,0}M\) is the bundle \( \Lambda^{1,0} M \) of complex 1-forms of type \((1,0)\). By (complex) duality, the canonical Cauchy-Riemann operator \( \overline{\partial}_X^{TM} \) defined in Proposition 9.3.1 induces a Cauchy-Riemann operator on \( \Lambda^{1,0} \), say \( \overline{\partial}_X^{\Lambda^{1,0}M} \), defined by

\[
(\overline{\partial}_X^{\Lambda^{1,0}M} \zeta, Z) = \overline{\partial}_X \langle \zeta, Z \rangle - \langle \zeta, \overline{\partial}_X^{TM} Z \rangle,
\]

for any \((1,0)\)-form \( \zeta \), any (real) vector field \( X \) and any vector field \( Z \) (in (9.3.3), \( \langle \cdot, \cdot \rangle \) denotes the (complex) duality between \((TM,J) \cong T^{1,0}M\) and...
$\Lambda^{1,0} M$, whereas $Z$ can be regarded indifferently as a real vector field or as a complex vector field of type $(1,0)$). We then infer:

\begin{equation}
\bar{\partial}_X^{1,0} \zeta = (\mathcal{L}_{X^{0,1}} \alpha)^{1,0} = (\iota_{X^{0,1}} d\zeta)^{1,0} = \iota_{X^{0,1}} \bar{\partial} \zeta,
\end{equation}

for any $(1,0)$-form $\zeta$, where $\bar{\partial} \zeta := (d\zeta)^{1,1}$ denotes the $J$-invariant part of $d\zeta$.

As a real vector bundle, $\Lambda^{1,0} M$ is naturally identified with $T^* M$, in the same way as $T^{1,0} M$ has been identified with $TM$, i.e. by identifying any $\alpha$ in $T^* M$ with its $(1,0)$-part $\alpha^{1,0} = \frac{1}{2} (\alpha + iJ\alpha)$ and any $\zeta$ in $\Lambda^{1,0} M$ with $2\Re(\zeta) = \zeta + \zeta^\tau$. This identification is also an identification of complex vector bundles of $\Lambda^{1,0} M$, equipped with its natural structure of complex vector bundle, with $(T^* M, -J)$; the latter can then be regarded as the complex dual of $(TM, J)$, via the (complex) duality

\begin{equation}
\langle \alpha, X \rangle_C := \alpha^{1,0}(X) = \frac{1}{2} (\alpha(X) - i\alpha(JX)),
\end{equation}

for any (real) 1-form $\alpha$ and any (real) vector field $X$.

In this setting, the canonical Cauchy-Riemann operator $\bar{\partial}^{\Lambda^{1,0}}$ is better denoted by $\bar{\partial}^{TM}$ and (9.3.4) then reads:

\begin{equation}
(\bar{\partial}_X^{TM} \alpha)(Z) = \frac{1}{2} X \cdot \alpha(Z) + \frac{1}{2} JX \cdot \alpha(JZ) - \alpha(\bar{\partial}_X^{TM} Z),
\end{equation}

for any (real) 1-form $\alpha$ and any two (real) vector fields $X, Z$, where $\bar{\partial}_X^{TM} Z$, itself viewed as a real vector field, can be replaced by any expression in the rhs of (9.3.2).

If $J$ is a part of an almost hermitian structure $(g, J, \omega)$, $(TM, J)$ comes equipped with a (fiberwise) hermitian inner product $h$ defined by $h(X, Y) = g(X^{1,0}, Y^{0,1}) = \frac{1}{2} (g(X, Y) - i\omega(X, Y))$. According to the general receipt explained in Section 1.6 — cf. in particular Proposition 1.6.1 — this hermitian inner product, together with the canonical Cauchy-Riemann operator (9.3.1), determines a canonical Chern connection $\nabla^2$, defined by

\begin{equation}
\nabla_X Z = \bar{\partial}_X^{TM} Z + \bar{\partial}_X^{TM} Z,
\end{equation}

by setting $\bar{\partial}_X^{TM} Z := \tau^{-1}(\bar{\partial}^{\Lambda^{1,0} M} \tau(Z))$, where $\tau$ denotes the hermitian duality from $(TM, J) \cong T^{1,0} M$ to $(T^* M, -J) \cong \Lambda^{1,0} M$. Notice that $\tau$ and its inverse, $\tau^{-1}$, are given by

\begin{equation}
\tau(X) = (X^\flat)^{1,0} = \frac{1}{2} (X^\flat + iJX^\flat), \quad \tau^{-1}(\zeta) = 2(\Re \zeta)^{\flat},
\end{equation}

for any $X$ in $TM$ and any $\zeta$ in $\Lambda^{1,0} M$, when $\tau$ is regarded as a (C-antilinear) map from $(TM, J)$ to $\Lambda^{1,0} M$, whereas $\tau$ and $\tau^{-1}$, regarded as maps from $(TM, J)$ to $(T^* M, -J)$ and from $(T^* M, -J)$ to $(TM, J)$, are simply the riemannian isomorphims $\flat$ and $\sharp$ defined in Section 1.2. We then have:

**Proposition 9.3.2.** For any almost-hermitian structure $(g, J, \omega)$, the canonical Chern connection $\nabla$ has the following expression

\begin{equation}
\nabla_X Z = 2\Re([X^{0,1}, Z^{1,0}]^{1,0}) + (2\Re(\iota_{X^{0,1}}(\bar{\partial}(Z^\flat)^{1,0}))^\flat
\end{equation}

\footnote{This connection is the second canonical hermitian connection as defined in A. Lichnerowicz’s book [136], cf. also [89].}
for any (real) vector fields $X$ and $Z$. Alternatively,

$$
(\nabla_X Z, Y) = \frac{1}{2} X \cdot (Z, Y) + \frac{1}{2} JX \cdot (Z, JY)
$$

(9.3.10)

$$
+ \frac{1}{4} \left( [X, Z] + [JX, JZ] + J[X, Z] - J[X, JZ], Y \right)
$$

$$
- \frac{1}{4} \left( [X, Y] + [JX, JY] + J[X, Y] - J[X, JY], Z \right),
$$

for any (real) vector fields $X, Y, Z$.

Proof. (9.3.9) is an immediate consequence of (9.3.1), (9.3.4) and (9.3.8). By using (9.3.6) and (9.3.2), we readily get (9.3.10). □

By using (1.23.4), we now check that the canonical Cauchy-Riemann operator $\bar{\partial}^{TM}$ can be expressed in terms of the Levi-Civita connection $D$ of $g$ as follows:

$$
\bar{\partial}^{TM}_X Z = \frac{1}{2} (D_X Z + JD_J X Z)
$$

(9.3.11)

$$
- \frac{1}{4} (D_{JZ} J + JD_{ZJ}) X + \frac{1}{4} (D_{JX} J - JD_{XJ}) Z.
$$

Similarly, from (9.3.4) we easily infer:

$$
(\partial^{TM}_X Z, Y) = \frac{1}{2} (D_X Z - JD_{JX} Z, Y)
$$

(9.3.12)

$$
- \frac{1}{4} (D_J X J + JD_{XJ}) Z, Y) + \frac{1}{4} (Z, (D_JY J + JD_{YJ}) X).
$$

For any almost-hermitian structure, the Chern connection $\nabla$ and the Levi-Civita connection $D$ are then related by

$$
(\nabla_X Z, Y) = (D_X Z, Y) - \frac{1}{2} (J(D_J X) Z, Y)
$$

(9.3.13)

$$
+ \frac{1}{4} \left( (D_{JY} J + JD_{YJ}) X, Z \right) - \frac{1}{4} \left( (D_{JX} J + JD_{XJ}) X, Y \right).
$$

In the almost Kähler case, which is the case of main interest in this section, this expression much simplifies thanks to the following lemma:

**Lemma 9.3.1.** An almost hermitian structure $(g, J, \omega)$ is almost Kähler if and only if $DJ$ and the Nijenhuis tensor $N$ satisfy the identities

$$
D_{JX} J = -J D_X J
$$

(9.3.14)

and

$$
(X, N(Y, Z)) + (Y, N(Z, X)) + (Z, N(X, Y)) = 0.
$$

(9.3.15)

Proof. By (1.1.5), if $d\omega = 0$, $DJ$ and $N$ are related by

$$
((D_J X) Y, Z) = 2(JX, N(Y, Z)),
$$

(9.3.16)

for any vector fields $X, Y, Z$. It readily follows that $DJ$ satisfy (9.3.14) — which is the opposite of the identity (1.1.6) which characterizes hermitian structures — and that $N$ satisfies the Bianchi identity (9.3.15). Conversely, if
(9.3.14) is satisfied, from (1.1.3) we get \( N(X, Y) = \frac{1}{2} J((D_Y J) X - (D_X J) Y) \); then, (9.3.15) and (9.3.14) again implies \( ((D_{JX} J) Z, Y) + ((D_J Z) JX, Y) + ((D_Y J) JX, Z) = d\omega(JX, Z, Y) = 0 \), for any vector fields \( X, Y, Z \). We then get \( d\omega = 0 \).

As a direct corollary, we obtain

**Proposition 9.3.3.** For any almost Kähler structure \((g, J, \omega)\) the canonical Cauchy-Riemann operator \( \partial^TM \) and the Chern connection \( \nabla \) are given by

\[
\partial^TM Z = \frac{1}{2} (D_X Z + J D_{JX} Z) - \frac{1}{2} J(D_X J) Z,
\]

and

\[
\nabla X Z = D_X Z - \frac{1}{2} J(D_X J) Z.
\]

If \( T^n \) denotes the torsion of \( \nabla \), we then have

\[
T^n = 0.
\]

**Proof.** (9.3.17) readily follows (9.3.11) and (9.3.14) and (9.3.18) is also an immediate consequence of (9.3.13) and (9.3.14). From (9.3.18), we infer:

\[
(T^n(X, Y) = -\frac{1}{2} (JD_X J) Y + \frac{1}{2} (DY J) X; \text{ by (1.1.3), this expression is equal to } N(X, Y) \text{ whenever (9.3.14) is satisfied.}
\]

**Remark 9.3.2.** Lemma 9.3.1 can be made more precise by considering the type decomposition of the (real) 3-form \( d\omega \), which splits as \( d\omega = (d\omega)^{2,1,1,2} + (d\omega)^{3,0,0,3} \). It can then be checked that \( (d\omega)^{2,1,1,2} = 0 \) if and only if \( DJ \) satisfies (9.3.14), whereas \( (d\omega)^{3,0,0,3} = 0 \) if and only if \( N \) satisfies the Bianchi identity (9.3.15).

Now, Proposition 9.3.3 has been derived by using (9.3.14) only, not (9.3.15): it then holds with the weaker assumption that \( (d\omega)^{2,1,1,2} = 0 \), i.e., this class of almost hermitian structures — one of the 16 classes appearing in the Gray-Hervella classification [92] — has been called (2, 1)-symplectic by S. Salamon [171].

For a general almost hermitian structure, the hermitian connection defined by the rhs of (9.3.18) is the connection obtained by orthogonal projection of the Levi-Civita connection \( D \) into the space of hermitian connections and has therefore been called the first canonical hermitian connection in [136]. Proposition 9.3.3 can then be rephrased by saying that for any almost Kähler structure, actually for any (2, 1)-symplectic almost hermitian structure, the first and second hermitian connections coincide. The converse is also true: an almost hermitian structure is (2, 1)-symplectic if and only if the first and second hermitian connections coincide. More generally, a 1-parameter family of hermitian connections, denoted by \( \nabla^t \) in [89], is canonically attached to any almost hermitian structure, where the first canonical hermitian connection is \( \nabla^0 \) and the second canonical hermitian connection, or Chern connection, is \( \nabla^1 \); it can then be shown that \( \nabla^0 = \nabla^1 \) if and only if \( \nabla^t = \nabla^0 \) for all \( t \), if and only if the structure is (2, 1)-symplectic, cf. [89] for details.

From (9.3.19), we infer that the Chern connection of an almost-Kähler structure — more generally of a (2, 1)-symplectic almost-hermitian structure
— is the (unique) connection which preserves the metric $g$ whose torsion is equal to the Nijenhuis tensor. For a general almost-hermitian structure this connection is not a hermitian connection, i.e. does not preserves $J$. It is easily checked that it does preserve $J$ if and only if the almost-hermitian structure is $(2,1)$-symplectic and is then equal to the Chern connection. Indeed, in general, a connection, $D^T$, which preserves $g$ and whose torsion is $T$ is uniquely defined and is determined by

$$(D^T_X Z, Y) = (D_X Z, Y) + \frac{1}{2} (T(X, Z), Y) - (T(Z, Y), X) - (T(Y, X), Z),$$

where $D$ denotes the Levi-Civita connection of $g$; if $T = N$, we infer that $D^T$ preserves $J$ is and only if

$$((D_X J)Z, Y) = -(N(X, Z), JY) + (N(Z, Y), JX) + (N(Y, X), JZ);$$

by (1.1.7), this condition is equivalent to (9.3.14).

## 9.4. The hermitian scalar curvature

For a general almost hermitian structure $(g, J, \omega)$, the Chern connection induces a hermitian connection on the anti-canonical bundle $K^*M = \Lambda^m((TM, J))$ — cf. Section 1.19. Since $K^*M$ is a complex line bundle, the curvature of the induced hermitian connection is of the form $i\rho^\nabla$, for some (closed) real 2-form $\rho^\nabla$, called the hermitian Ricci form. The hermitian scalar curvature, $s^\nabla$, is then defined as the normalized trace of $\rho^\nabla$ with respect to $\omega$, namely by

$$s^\nabla = 2\Lambda(\rho^\nabla).$$

The normalization is chosen in order that $s^\nabla$ coincide with the riemannian scalar curvature $s$ in the Kähler case ($\rho^\nabla$ then coincides with the Ricci form $\rho$ defined in Section 1.19).

In order to relate the hermitian scalar curvature $s^\nabla$ to the scalar curvature $s$, at least in the almost-Kähler setting, it is convenient to introduce the 2-form $R(\omega)$, image of the Kähler form by the (riemannian) curvature operator, often called the $*$-Ricci form. We similarly introduce the $*$-scalar curvature $s^*$, defined by

$$s^* = 2\Lambda(R(\omega)).$$

Again the normalization is chosen so that $s^*$ coincides with $s$ in the Kähler case, when $R(\omega)$ itself coincides with the Ricci form $\rho$. In general however, $R(\omega)$ is not closed, nor even $J$-invariant.

### Proposition 9.4.1 (V. Apostolov-T. Draghici [8]).

For any almost Kähler structure $(g, J, \omega)$, the $*$-Ricci form $R(\omega)$, the hermitian Ricci form $\rho^\nabla$ and the riemannian Ricci tensor $r$ are related by

$$\rho^\nabla_{X,Y} = R(\omega)_{X,Y} - \frac{1}{4} \text{tr}(JD_X J \circ D_Y J),$$

and

$$R(\omega)_{X,Y} = \frac{1}{2} (r(JX, Y) - r(X, JY)) + \frac{1}{2} ((D^* DJ) X, Y).$$
The hermitian scalar curvature $s^\nabla$, the $\ast$-scalar curvature $s^\ast$ and the riemannian scalar curvature $s$ are then related by

\begin{equation}
(9.4.5) \quad s^\nabla = s + \frac{1}{2} |DJ|^2 = s^\ast - \frac{1}{2} |DJ|^2 = \frac{1}{2} (s + s^\ast).
\end{equation}

**Proof.** For any almost hermitian structure $(g, J, \omega)$, any hermitian connection $\nabla$ is of the form $\nabla = D + \eta$, where $\eta$ is a 1-form with values in the bundle $A(TM)$ of skew-symmetric endomorphisms of $TM$ and such that $[\eta_X, J] = - DJ$ for any vector field $X$. The curvature $R^\nabla$ is then given by

\begin{equation}
(9.4.6) \quad R^\nabla_{X,Y} = R_{X,Y} - (d^D \eta)_{X,Y} - [\eta_X, \eta_Y] = R_{X,Y} - (d^D \eta)_{X,Y} + [\eta_X, \eta_Y]
\end{equation}

(the operators $d^\nabla$ and $d^D$, where $\nabla$ and $D$ actually stands for the induced connection on $A(TM)$, have been defined in (1.6.2); the general expression (9.4.6) for the curvature of $\nabla = D + \eta$, or $D = \nabla - \eta$, is then an easy consequence of (9.4.6)).

The corresponding Ricci form, $\rho^\nabla$ — defined like in the case that $\nabla$ is the Chern connection, i.e. by $\rho^\nabla = \frac{1}{2} \text{tr}(-J \circ R^\nabla) —$ is then given by

\begin{equation}
(9.4.7) \quad \rho^\nabla_{X,Y} = R(\omega)_{X,Y} + \frac{1}{2} (d(\text{tr}(J \circ \eta)))_{X,Y} - \frac{1}{2} \text{tr}(J[\eta_X, \eta_Y]).
\end{equation}

When $\nabla$ is the first canonical hermitian connection, as defined in Remark 9.3.2, we have $\eta_X = - \frac{1}{2} DJ_X J$, which is trace-free, and the above identity reduces to

\begin{equation}
(9.4.8) \quad \rho^\nabla_{X,Y} = R(\omega)_{X,Y} - \frac{1}{4} \text{tr}(JD_X J \circ D_Y J).
\end{equation}

This holds for any almost hermitian structure when $\rho^\nabla$ stands for the Ricci form of the first canonical hermitian connection. In the almost Kähler case, the latter is equal to the Chern connection and we thus get (9.4.3). We now have:

\begin{equation}
(9.4.9) \quad R(\omega)_{X,Y} = \frac{1}{2} \sum_{i=1}^n (R_{X,Y} e_i, J e_i)
\end{equation}

\begin{align*}
&= - \frac{1}{2} \sum_{i=1}^n ((R_{Y,e_i} X, J e_i) + (R_{e_i, X Y}, J e_i)) \\
&= \frac{1}{2} \sum_{i=1}^n ((J R_{Y,e_i} X, e_i) + (J R_{e_i, X Y}, e_i)) \\
&= \frac{1}{2} \sum_{i=1}^n ((R_{Y,e_i} J X, e_i) + (R_{e_i, X J Y}, e_i)) \\
&- \frac{1}{2} \sum_{i=1}^n ((R_{Y,e_i} J X, e_i) + (R_{e_i, X J Y}, e_i))
\end{align*}

where $\{e_i\}, i = 1, \ldots, n$, stands for any auxiliary local orthonormal frame. From the tautological commutation formula

\begin{equation}
(9.4.10) \quad [R_{X,Y}, J] = R_{X,Y} J = (D^2 J)_{Y,X} - (D^2 J)_{X,Y},
\end{equation}

compute
which holds in the general almost hermitian case, we then infer:

\[ R(\omega)_{X,Y} = \frac{1}{2} (r(JX,Y) - r(X,JY)) \]

(9.4.11)

\[ + \frac{1}{2} \sum_{i=1}^{n} (((D_{e_i}^2 X)Y, e_i) - ((D_{e_i}^2 Y)X, e_i)) \]

\[ - \frac{1}{2} \sum_{i=1}^{n} (((D_{e_i}^2 X)Y, e_i) - ((D_{e_i}^2 Y)X, e_i)). \]

Now, since \((g, J, \omega)\) is almost Kähler, we have

\[ \delta J = - \sum_{i=1}^{n} (D_{e_i} J)(e_i) = 0, \]

(9.4.12)

and the Bianchi identity

\[ ((D_X J)Y, Z) + ((D_Y J)Z, X) + ((D_Z J)X, Y) = 0. \]

(9.4.13)

From (9.4.12) we infer that the last line of the rhs of (9.4.11) vanishes and from (9.4.13) we infer that \( \sum_{i=1}^{n} (((D_{e_i}^2 X)Y, e_i) - ((D_{e_i}^2 Y)X, e_i)) \) is equal to \(- \sum_{i=1}^{n} ((D_{e_i}^2 X)J)X, Y) = ((D^*D)J)X, Y); we thus get (9.4.4). Then, (9.4.5) readily follows from (9.4.4) and (9.4.3) (note however that we used the convention \(|J|^2 = |\omega|^2 = -\frac{1}{2} \text{tr}(J \circ J), \) cf. Section 1.3).

**Remark 9.4.1.** In [9], (9.4.4) is deduced from the following general Weitzenböck-Bochner formula

\[ \Delta \psi = D^*D\psi + 2R(\psi) - (r(\Psi \cdot, \cdot) - r(\cdot, \Psi \cdot)), \]

(9.4.14)

which holds for any 2-form \( \psi \) and any riemannian metric \( g \), when \( D \) stands for the Levi-Civita connection, \( \Delta \) for the riemannian Laplacian and \( \Psi \) for the skew-symmetric endomorphism defined by \( g(\Phi X, Y) = \psi(X, Y) \). In the almost Kähler case, the Kähler form \( \omega \) is harmonic and (9.4.4) is then a direct application of (9.4.14).

Note that, conversely, the proof of Proposition 9.4.1 can be readily extended to give a proof of (9.4.14) as well.

**Remark 9.4.2.** In the almost Kähler case, the hermitian scalar curvature is a natural substitute of the riemannian scalar curvature for many purposes. In particular, the total hermitian scalar curvature \( S^\nabla := \int_M s^\nabla v_g \) is constant on \( AC_{\omega_0} \) and the critical points in \( AC_{\omega_0} \) of the hermitian Calabi functional

\[ C^\nabla(J) := \int_M (s^\nabla)^2 v_g, \]

(9.4.15)

are again the \( J \) such that \( s^\nabla \) is the momentum of a hamiltonian Killing vector fields for the corresponding metric \( g \). These facts are mentioned in [9] and will be proven in the next section as a consequence of the Mohsen formula.
9.5. The Mohsen formula

We now restrict our attention to the case when the almost complex structure $J$ belongs to the space $\mathcal{AC}_{\omega_0}$ where $\omega_0$ is some fixed symplectic form on $M$, so that $(g, J, \omega_0)$ is an almost Kähler with $g = \omega_0(\cdot, J \cdot)$. We denote by $\nabla^J$ the corresponding Chern connection on $(TM, J)$ and by $\nabla^{K^*_J}$ the induced connection on the corresponding anti-canonical bundle $K^*_J$; note that $K^*_J$ is a complex vector subbundle of $\Lambda^m TM \otimes \mathbb{C}$ dependent of $J$. Our main goal is to compute the first variation of $\nabla^{K^*_J}$ when $J$ moves in $\mathcal{AC}_{\omega_0}$ in the following sense. For any $J$ in $\mathcal{AC}_{\omega_0}$, any $\dot{J}$ in $T_J \mathcal{AC}_{\omega_0}$ is the tangent vector to the curve

\[ J_t := \exp(ta) J \exp(-ta), \]

where $a = \frac{1}{2} J \dot{J}$ is a well-defined section of $\text{End}(TM, \omega_0)$, cf. Section 9.2. For simplicity, we shall denote $\exp(ta)$ by $\gamma_t$; then, $\gamma_t$ is an isomorphism from $(TM, J)$ to $(TM, J_t)$ and induces an isomorphism, still denoted by $\gamma_t$, from $K^*_J$ to $K^*_J$. Since $\gamma_t$ pointwise preserves $\omega_0$, it is actually an isomorphism of hermitian line bundles so that the connection $\tilde{\nabla}^{K^*_J}$ defined above, (9.5.2) can be rewritten as

\[ \tilde{\nabla}^{K^*_J} \xi_t = \alpha_t \otimes J \xi_t, \]

where $\alpha_t$ is a real (local) 1-form, called the connection 1-form of $\tilde{\nabla}^{K^*_J}$ with respect to $\xi_t$. The hermitian Ricci form $\rho^{\nabla^J}$ and the hermitian scalar curvature $s^{\nabla^J}$ of $J_t$ are then locally given by

\[ \rho^{\nabla^J} = -d\alpha, \quad s^{\nabla^J} = -2\Lambda d\alpha. \]

By using the connection $\nabla^{\nabla^J}$ defined above, (9.5.2) can be rewriten as

\[ \nabla^{\nabla^J} \xi = \alpha_t \otimes J \xi \]

so that $\alpha_t$ now appears as the connection 1-form of the (moving) connection $\nabla^{\nabla^J}$ with respect to $\xi_t$. The first variation along $\dot{J}$ of the Chern connection of the anti-canonical bundle is given by

\[ \dot{\alpha} = \frac{1}{2} (\delta \dot{J})^\flat, \]

where the codifferential $\delta$ is relative to the metric $g = \omega_0(\cdot, J \cdot)$ and the vector field $\delta \dot{J}$ is then defined by

\[ \delta \dot{J} = - \sum_{i=1}^n (D_{e_i} J)(e_i), \]
for any auxiliary orthonormal frame \( \{ e_1, \ldots, e_n \} \), where \( D \) stands for the Levi-Civita connection of \( g \).

**Proof.** A hermitian orthonormal (local) frame for \((TM, J)\) is a system of \( m \) local vector fields \( Z_1, \ldots, Z_m \) such that
\[
h(Z_j, Z_k) := \frac{1}{2} (g(Z_j, Z_k) - \imath \omega_0(Z_j, Z_k)) = \delta_{jk}, \text{ i.e. } g(Z_j, Z_k) = 2 \delta_{jk} \text{ and } \omega_0(Z_j, Z_k) = 0, \text{ for any } j, k = 1, \ldots, m \text{ (recall that the (real) dimension of } M \text{ is } n = 2m). \]
Fix such a local frame; the corresponding \( \alpha_t \) has then the following expression:

\[
\alpha_t(X) = -i \sum_{j=1}^{m} h(\nabla^J_X Z_j, Z_j)
\]
\[
= -i \sum_{j=1}^{m} (h(\nabla^J_X \gamma^{-1}_t Z_j, Z_j) - h(Z_j, \nabla^J_X \gamma^{-1}_t Z_j))
\]
\[
= -i \sum_{j=1}^{m} h(\gamma^{-1}_t \partial^{(TM, J)}_X \gamma^{-1}_t Z_j, Z_j) - h(Z_j, \gamma_t \partial^{(TM, J)}_X \gamma^{-1}_t Z_j)
\]
\[
= -\omega_0(\gamma^{-1}_t \partial^{(TM, J)}_X \gamma^{-1}_t Z_j, Z_j).
\]

(9.5.7)

By using (9.3.2) and the identity \( J_t \gamma_t = \gamma_t J \), we infer

\[
\alpha_t = \frac{1}{4} \sum_{j=1}^{m} (\omega_0(\gamma^{-1}_t (\mathcal{L}_{\gamma_t Z_j} J_t)(X), Z_j) + \omega_0(\gamma^{-1}_t J_t (\mathcal{L}_{\gamma_t Z_j} J_t)(X), Z_j)
\]
\[
= \frac{1}{4} \sum_{j=1}^{m} (\omega_0(\gamma^{-1}_t (\mathcal{L}_{\gamma_t Z_j} J_t)(X), Z_j) + \omega_0(J_{\gamma^{-1}_t} (\mathcal{L}_{\gamma_t Z_j} J_t)(X), Z_j)
\]
\[
= -\frac{1}{2} \sum_{k=1}^{n} g(\gamma^{-1}_t (\mathcal{L}_{\gamma_t e_k} J_t)X, e_k)
\]

(9.5.8)

where \( \{ e_k \}, k = 1, \ldots, n = 2m \) stands for the \( g \)-orthonormal frame \( \{ Z_1, \ldots, Z_m, JZ_1, \ldots, JZ_m \} \). Recall that \( \dot{\alpha} \) is obtained by taking the derivative of the above expression with respect to \( t \) at \( t = 0 \). For that it is convenient to previously transform this expression by replacing the Lie derivative by the Levi-Civita connection \( D \) of the (fixed) metric \( g = \omega_0(\cdot, J\cdot) \).
by using (1.23.4), so that:

\[
\alpha_t = -\frac{1}{2} \sum_{k=1}^{n} g(\gamma_t^{-1}(D_{\gamma_t e_k} J_t)(X), e_k) - \frac{1}{2} \sum_{k=1}^{n} g(\gamma_t^{-1}[J_t, D(\gamma_t e_k)])(X), e_k) \\
= -\frac{1}{2} \sum_{k=1}^{n} g((D_{e_k} J_t)(X), e_k) \quad \quad (9.5.9)
\]

\[
- \frac{1}{2} \sum_{k=1}^{n} g(J \gamma_t^{-1} D_X(\gamma_t e_k), e_k) \\
- \frac{1}{2} \sum_{k=1}^{n} g(\gamma_t^{-1} D_{J_t X}(\gamma_t e_k), e_k).
\]

We thus get

\[
\dot{\alpha} = -\frac{1}{2} \sum_{k=1}^{n} g((D_{e_k} J_t)(X), e_k) \\
+ \frac{1}{2} \sum_{k=1}^{n} g(J(D_X a)e_k, e_k) \quad \quad (9.5.10)
\]

+ \frac{1}{2} \sum_{k=1}^{n} (g((D_J X a)e_k, e_k) - g(D_J e_k, e_k)).

Since \( \dot{J} \) is symmetric, the first term of the rhs in the above expression is equal to \( \frac{1}{2} g(\delta \dot{J}, X) \), whereas the other terms are 0: this is because \( e_k \) is of norm 1, so that \( g(D_a X e_k, e_k) = 0 \), and \( a \) and \( J a \) are trace-free, as they both anticommute to \( J \). We thus get (9.5.5).

As a direct consequence of the Mohsen formula, we get

\begin{proposition}
For any almost complex structure \( J \) in \( AC_{\omega_0} \), the derivative of the hermitian Ricci form \( \dot{\rho}^\nabla \) and of the hermitian scalar curvature \( \dot{s}^\nabla \) along any \( \dot{J} \) in \( T_J AC_{\omega_0} \) is given by

\[
\dot{\rho}^\nabla = -\frac{1}{2} d \delta (\dot{J})^b, \quad \dot{s}^\nabla = -\Lambda d(\delta \dot{J})^b = -\delta J(\delta \dot{J})^b. \quad \quad (9.5.11)
\]

\end{proposition}

\begin{proof}
From (9.5.3), we get

\[
\dot{\rho}^\nabla = -d\dot{\alpha}, \quad \dot{s}^\nabla = -2\Lambda d\dot{\alpha}. \quad \quad (9.5.12)
\]

The given expression of \( \dot{\rho}^\nabla \) and the first given expression of \( \dot{s}^\nabla \) are then a direct consequence of the Mohsen formula (9.5.5). The second given expression of \( \dot{s}^\nabla \) in (9.5.11) follows from the Kähler identity \([\Lambda, d] = -\delta \), which, we recall, holds in the almost Kähler case, cf. Section 1.14.

\end{proof}

\begin{remark}
If \( J \) belongs to \( C_{\omega_0} \) then \((g, J, \omega_0)\) is Kähler. From (9.5.6), we infer \( J \delta \dot{J} = \delta (J \dot{J}) \) and the above formula for \( \dot{s}^\nabla \) can then be read as

\[
\dot{s}^\nabla = -\delta(\delta (J \dot{J}))^b = \delta \delta \dot{g}. \quad \quad (9.5.13)
\]
\end{remark}
where \( \dot{g} \) denotes the derivative of \( g_t = \omega_0(\cdot, J_t) \), hence given by \( \dot{g}(X,Y) = -g(JX,Y) \). Since the hermitian scalar curvature of a Kähler manifold coincides with the riemannian scalar curvature this formula should appear as a special case of the general formula (5.4.4) of the variation of the riemannian scalar curvature, at least when \( \dot{J} \) belongs to the subspace \( T_J \mathcal{AC} \) and this is actually the case. Indeed, if \( \dot{J} \) belongs to \( T_J \mathcal{AC} \), even to \( T_J \mathcal{AC}_0 \), the corresponding variation \( \dot{g} \) of the metric is trace free and \( J \)-antilinear, hence orthogonal to the (riemannian) Ricci tensor \( r \), which is \( J \)-invariant as \( (g,J) \) is Kähler. The rhs of (5.4.4) then reduces to the rhs of (9.5.13).

From (9.5.11), we infer that the Ricci forms, \( \rho^\nabla \) and \( \rho^\nabla_0 \), of any two elements, \( J \) and \( J_0 \), of \( \mathcal{AC}_0 \) are related by

\[
(9.5.14) \quad \rho^\nabla - \rho^\nabla_0 = -\frac{1}{2} d\left( \int_0^1 (\delta g_t \dot{J}_t)^{\omega_t} \, dt \right),
\]

for any curve \( J \) which joins \( J_0 \) to \( J \) in \( \mathcal{AC}_0 \). In particular, \( \rho^\nabla \) and \( \rho^\nabla_0 \) belong to the same de Rham class in \( H^2_{dR}(M, \mathbb{R}) \). This fits with the fact that all almost-complex structures in \( \mathcal{AC}_0 \) belong to a same deformation class — as they are sections of a bundle with contractible fibers, cf. Section 9.2 — hence have the same Chern class (called the Chern class of the symplectic manifold \( (M, \omega_0) \)).

We also infer that the total hermitian scalar curvature \( S^\nabla = \int_M s^\nabla v_g \) is constant on \( \mathcal{AC}_0 \). Indeed, by (9.5.11), we get,

\[
(9.5.15) \quad S^\nabla - S^\nabla_0 = -\int_M \Lambda d\left( \int_0^1 (\delta g_t \dot{J}_t)^{\omega_t} \, dt \right) v_g = 0
\]

Moreover, we have the following almost Kähler generalization of Theorem 3.2.1, already mentioned in [9]:

**Proposition 9.5.3.** An element \( J \) of \( \mathcal{AC}_0 \) is a critical point for the hermitian Calabi functional (9.4.15) defined on \( \mathcal{AC}_0 \) if and only if the hamiltonian vector field \( \mathbb{K}^\nabla := \text{grad}_{\omega_0} s^\nabla = J \text{grad}_g s^\nabla \) is Killing with respect to the metric \( g = \omega_0(\cdot, J \cdot) \).

**Proof.** From (9.5.11) we get

\[
(9.5.16) \quad dC_j^\nabla(J) = 2 \int_M s^\nabla \delta^c(\delta \dot{J})^{\omega_0} v_g = 2 \langle D\text{grad}_{\omega_0} s^\nabla, J \rangle,
\]

for any \( \dot{J} \) in \( T_J \mathcal{AC}_0 \). Recall that the latter is the space of \( g \)-symmetric, \( J \)-anti-invariant endomorphisms of \( TM \). It follows that \( J \) is a critical point if and only if the symmetric, \( J \)-anti-invariant part of \( D\text{grad}_{\omega_0} s^\nabla \) vanishes identically. By (9.2.2), for any hamiltonian vector field \( Z \), the symmetric part of \( DZ \) is equal to \( -2J L_2 J \), hence already \( J \)-invariant. We infer that \( J \) is critical if and only if \( D\text{grad}_{\omega_0} s^\nabla \) is skew-symmetric, if and only if \( \text{grad}_{\omega_0} s^\nabla \) is a (hamiltonian) Killing vector field.

**9.6. Hamiltonian action of Ham\((M, \omega_0)\) on \( \mathcal{AC}_0 \)**

The following fact has been first observed by A. Fujiki [84] in the Kähler case — cf. Remark 9.2.1 — and by S. Donaldson [73] in the general almost-kähler case.
Theorem 9.6.1. For any compact symplectic manifold \((M, \omega_0)\), the action of \(\text{Ham}(M, \omega_0)\) on the space \(\mathcal{AC}_{\omega_0}\) of all \(\omega_0\)-compatible almost-structure is Hamiltonian, with momentum \(\mu\) given by

\[
\mu^Z(J) = -\int_M f s_g^\nabla v_g = -\int_M f s_g^\nabla \frac{\omega_0^m}{m!},
\]

for any \(Z = \text{grad}_\omega f\) in \(\mathfrak{ham}\), where \(s_g^\nabla\) denotes the Hermitian scalar curvature of the almost Kähler structure \((g = \omega_0(\cdot, J\cdot), J, \omega_0)\).

Proof. Since the action of \(\text{Sp}(M, \omega_0)\) on \(\mathcal{AC}_{\omega_0}\) preserves \(\kappa\) and \(\mathcal{AC}_{\omega_0}\) is contractible, we know that for any \(Z\) in \(\mathfrak{sp}\), the induced vector field \(\dot{Z}\) admits a momentum, \(\mu^Z\), defined, up to an additive constant, by

\[
\dot{t}_Z\kappa = -d\mu^Z.
\]

By (9.2.2) \(\dot{Z}J\) can be written as

\[
\dot{Z}J = -L_Z J = -J(DZ + (DZ)^*),
\]

for any \(Z\) in \(\mathfrak{sp}\) and any \(J\) in \(\mathcal{AC}_{\omega_0}\). From (9.2.11) we then infer

\[
\kappa(\dot{Z}J, J) = (DZ, J) = (Z, \delta J).
\]

where \(\langle \cdot, \cdot \rangle\) denotes the global inner product with respect to \(g\) (recall that \(J\) is symmetric with respect to \(g\), so that \(\langle J, DZ \rangle = \langle J, (DZ)^* \rangle\)). This holds for any \(Z\) in \(\mathfrak{sp}\). In the case when \(Z\) belongs to the Lie subalgebra \(\mathfrak{ham}\), then \(Z = \text{grad}_\omega f^Z = J \text{grad}_g f^Z\), where \(f^Z\) denotes the moment of \(Z\) with respect to \(\omega_0\) normalized by \(\int_M f^Z \omega_0^m = 0\) (note that \(f^Z\) only depends on \(Z\) and \(\omega_0\)). By using the expression of the variation of Hermitian scalar curvature \(s^\nabla\) given by (9.5.11), (9.6.4) can then be rewritten as:

\[
\kappa(\dot{Z}J, J) = \langle df^Z, \delta J \rangle = \langle f^Z, \delta(J^\flat) \rangle = \int_M f^Z s^\nabla(J) \frac{\omega_0^m}{m!}.
\]

Since \(f^Z\) only depends on \(Z\) and \(\omega_0\), we thus get

\[
\kappa(\dot{Z}J, J) = -d\mu^Z_J(J),
\]

by setting:

\[
\mu^Z(J) := -\int_M f^Z s^\nabla \frac{\omega_0^m}{m!}.
\]

It remains to check that \(\mu^Z\) defined by (9.6.7) determines a momentum map for the action of \(\text{Ham}(M, \omega_0)\), i.e. an equivariant map from \(M\) to the dual of the Lie algebra \(\mathfrak{ham}\), meaning that

\[
\mu^{\Phi \cdot J}(\Phi \cdot Z) = \mu^Z(J),
\]

for any \(Z\) in \(\mathfrak{ham}\), any \(J\) in \(\mathcal{AC}_{\omega_0}\), and any \(\Phi\) in \(\text{Ham}(M, \omega_0)\). This is because \(\Phi\) transports \((g, J, \omega_0)\) to \((\Phi \cdot g, \Phi \cdot J, \Phi \cdot \omega_0 = \omega_0)\), hence the anti-canonical bundle of \(J\) with its Chern connection \(\nabla^K_J\) to the anti-canonical bundle of \(\Phi \cdot J\) with its Chern connection \(\nabla^K_{\Phi \cdot J} = \Phi \cdot \nabla^K_J\). Similarly \(f^{\Phi \cdot Z} = \Phi \cdot f^Z\), so that the equivariance condition (9.6.8) immediately follows from (9.6.7). □
In view of Section 9.8 below, Theorem 9.6.1 can be given a more general formulation by fixing a connected, compact subgroup, \( G \), of \( \text{Ham}_0(M, \omega_0) \) and by replacing \( \mathcal{AC}_{\omega_0} \) by the space, \( \mathcal{AC}^G_{\omega_0} \), of \( G \)-invariant elements of \( \mathcal{AC}_{\omega_0} \). Denote by \( \text{Ham}_G(M, \omega_0) \) the (identity component of the) normalizer of \( G \) in \( \text{Ham}_0(M, \omega_0) \), and by \( \text{ham}_G \) its Lie algebra. Then, \( \text{ham}_G \) is the Lie algebra of elements \( X \) of \( \text{ham} \) such that \([X, Z]\) belongs to \( g \) whenever \( Z \) belongs to \( g \). The quotient \( \text{Ham}_G(M, \omega_0)/G \) acts on \( \mathcal{AC}^G_{\omega_0} \) and preserves the Fujiki Kähler structure. Denote by \( F^G_{\omega_0} \) the space of momenta — including constants — of elements of the Lie algebra, \( g \), of \( G \) in \( \text{ham}_G \), and by \( \Pi^G_{\omega_0} \) the orthogonal projection from \( C^\infty(M) \) to \( \Pi^G_{\omega_0} \), with respect to the symplectic volume form \( \omega_0^m \). For any \( J \) in \( \mathcal{AC}^G_{\omega_0} \), we then define the reduced hermitian scalar curvature \( s_G^G \) by

\[
\text{reduced-scalherm} \quad (9.6.9) \quad s_G^G = s_G^G + \Pi^G_{\omega_0}(s_f).
\]

we then have

\[
\text{momentuhmodified} \quad (9.6.10) \quad \mu_G^Z(J) = -\int_M f s_G^G \omega_0^m/m!,
\]

for any \( Z = \text{grad}_{\omega_0} f \) in \( \text{ham}_G / g \) and for any \( J \) in \( \mathcal{AC}^G_{\omega_0} \), where \( g = \omega_0(\cdot, J) \) stands for the associated riemannian metric.

**Proof.** This is a direct consequence of Theorem 9.6.1: the dual of the Lie algebra \( \text{ham}_G / g \) is the subspace of elements of \( (\text{ham}_G)^* \) which vanish on \( g \). Then, (9.6.10) readily follows from (9.6.1).

**Remark 9.6.1.** By Mohsen formula (9.5.5), (9.6.4) can be rewritten as

\[
\text{kappamohsen} \quad (9.6.11) \quad \kappa(\hat{Z}_J, \hat{J}) = 2 \int_M \hat{\alpha}(Z) v_g.
\]

For any chosen \( J_0 \) in \( \mathcal{AC}_{\omega_0} \) and any \( Z \) in \( \mathfrak{sp} \), the momentum, \( \mu^{\hat{Z}, J_0} \), of \( \hat{Z} \) which vanishes at \( J_0 \) is obtained at \( J \) by integrating (9.6.11) along any curve in \( \mathcal{AC}_{\omega_0} \) which joins \( J_0 \) to \( J \). By choosing the “geodesic” \( J_t = \exp(ta)J_0 \exp(-ta) \), where \( a \) is defined as in Lemma 9.2.2, we infer that \( \mu^{\hat{Z}, J_0} \) is given by

\[
\text{rem-mohsen} \quad (9.6.12) \quad \mu^{\hat{Z}, J_0}(J) = -2 \int_0^1 (\int_M \hat{\alpha}(Z) v_g) dt = -2 \int_M \psi^{J_0, J}(Z) v_g,
\]

where \( \psi^{J_0, J} \) is the real 1-form on \( M \) defined by

\[
\text{kappamohsen} \quad (9.6.13) \quad \exp(-a) \nabla^K_0 \exp(a) - \nabla^K_0 = \psi^{J_0, J} \nabla_0.
\]

In the case that \( Z = \text{grad}_{\omega_0} f \hat{Z} \) belongs to \( \text{ham} \), we infer as before that

\[
\text{rem-mohsen} \quad (9.6.14) \quad \mu^{\hat{Z}, J_0}(J) = -\langle s^0, f \hat{Z} \rangle + \langle s^0, f \hat{Z} \rangle
\]

where \( s^0 \) stands for the hermitian scalar curvature of \( \langle \omega_0 = \omega_0(\cdot, J_0) \rangle \). The set of momenta \( \mu^{\hat{Z}, J_0} \) does not form an equivariant momentum map by itself, as the chosen \( J_0 \) is not invariant, but in this case we nevertheless get an equivariant momentum map, equal to \( \mu^Z \), by simply substracting the constant \( \langle s^0, f \hat{Z} \rangle \) from each individual momentum \( \mu^{\hat{Z}, J_0} \). It is doubtful
however that we can arrange to obtain an equivariant momentum map for the action of whole group $\text{Sp}(M, \omega_0)$ on $\mathcal{A}C_{\omega_0}$.

**Remark 9.6.2.** It has been pointed out to me by V. Apostolov, the interpretation of the hermitian scalar curvature as a momentum map sheds a new light on Proposition 9.5.3 because of the following general fact. Let $(\mathcal{M}, g, J, \omega)$ be a Kähler manifold acted on in a hamiltonian way by a Lie group $G$, whose Lie algebra is denoted by $\mathfrak{g}$. Let $\mu : M \to \mathfrak{g}^*$ be the corresponding momentum map. By definition $\mu$ is $G$-equivariant and satisfies $\langle d\mu_\omega (X), a \rangle = -\omega_J (\hat{a}, X) = g_x (JX, \hat{a})$, for any $a$ in $\mathfrak{g}$, where $\hat{a}$ stands for the induced vector field on $M$ and $(\cdot, \cdot)$ her denotes the natural pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. Suppose that $\mathfrak{g}$ admits a $G$-equivariant inner product, which realizes $\mathfrak{g}$ as a subspace of the dual $\mathfrak{g}^*$, and suppose moreover that the image of $\mu$ belongs to $\mathfrak{g}$. We then define the square norm of $\mu$ by $|\mu|^2 (x) := \langle \mu(x), \mu(x) \rangle$, and the differential of $|\mu|^2$ at $x$ is then given by

$$
(9.6.15) \quad d|\mu|^2 (X) = 2 (d\mu_\omega (X), \mu(x)) = g_x (JX, \mu(x))
$$

for any $X$ in $T_x M$. We conclude that $x$ is a critical point of the function $|\mu|^2$ if and only if the condition

$$
(9.6.16) \quad \mu(x)(x) = 0
$$

is satisfies (cf. e.g. [117]). The above argument holds in an infinite-dimensional setting, in particular in the case that $\mathcal{M}$ is $\mathcal{A}C_{\omega_0}$, equipped with its natural Kähler structure $(\mathfrak{g}, J, \kappa)$, and $G$ here stands for the “Lie group” $\text{Ham}(M, \omega_0)$. According to Theorem 9.6.1 this action is hamiltonian, with momentum map $\mu(J) = s^V$ viewed as an element of $\text{Ham}^*$ via the duality $(s^V, f) = \int_M s^V f \mu_g$ for any $f$ in $\text{Ham}$ when the latter is identified with the space of all (smooth) real functions $f$ such that $\int_M f \omega_0^n = 0$. Now, the pairing $(f_1, f_2) := \int_M f_1 f_2 \nu_g$ induces an invariant inner product on $\text{Ham}$. By (tacitly) replacing $s^V$ by $s^V - \frac{s^V}{\omega_0}$ — this is innocuous as $s^V$ is constant on $\mathcal{A}C_{\omega_0}$ — $\mu(J)$ can then be considered as an element of the Lie algebra $\text{Ham}$ and the hermitian Calabi functional is then identified with the square norm $|\mu|^2$ as defined above. The critical points in $\mathcal{A}C_{\omega_0}$ are then characterized by (9.6.16). In the current case, we have that $\mu(J)(J) = -\mathcal{L}_{\text{grad}_{\omega_0} s^V} J$, so that condition (9.6.16) simply expresses the fact that $\text{grad}_{\omega_0} s^V$ preserves $J$. Since $\text{grad}_{\omega_0} s^V$ already preserves $\omega_0$, this amounts to saying that it is a Killing vector field with respect to $g$. We thus recover Proposition 9.5.3.

### 9.7. A symplectic Futaki invariant

Let $G$ be a compact Lie group acting in an effective, hamiltonian way on a connected, compact symplectic manifold $(M, \omega_0)$. As before, denote by $\mathcal{A}C_{\omega_0}^G$ the space of $G$-invariant $\omega_0$-compatible almost-complex structures on $M$.

For any $a$ in the Lie algebra $\mathfrak{g}$ of $G$, denote by $Z^a$ the induced vector field on $M$ and by $f^a$ the momentum of $Z^a$ relatively to $\omega_0$, so that $Z^a = \text{grad}_{\omega_0} f^a$. We assume that $f^a$ is normalized by $\int_M f^a \omega_0^n = 0$; it is the uniquely defined.
For any $J$ in $\mathcal{A}_{\omega_0}$, denote, as usual, by $g = \omega_0(\cdot, J)$ the corresponding riemannian metric, by $\nabla$ the corresponding Chern connection and by $s^\nabla$ the corresponding hermitian scalar curvature. We then have

**Proposition 9.7.1.** For any $a$ in $\mathfrak{g}$, the integral $\int_M f^a s^\nabla \frac{\omega_0}{m!}$ is independent of $J$ on any connected component of $\mathcal{A}_{\omega_0}^G$.

**Proof.** By (9.6.1), we have

$$\int_M f^a s^\nabla \frac{\omega_0}{m!} = -\mu \hat{z}^a(J),$$

where $\hat{Z}^a$ denotes the induced vector field on $\mathcal{A}_{\omega_0}^G$, defined by $\hat{Z}^a_j = -\mathcal{L}_{\hat{Z}^a} J$, and $\mu \hat{z}^a$ denotes the momentum of $\hat{Z}^a$ relatively to the canonical symplectic form $\kappa$ of $\mathcal{A}_{\omega_0}$. For any $J$ in $\mathcal{A}_{\omega_0}$ and any $\hat{J}$ in $T_J \mathcal{A}_{\omega_0}$, the variation of $\int_M f^a s^\nabla \frac{\omega_0}{m!}$ along $\hat{J}$ is then equal to $-d\mu \hat{z}^a(\hat{J}) = \kappa(\hat{Z}^a, \hat{J})$, cf. (9.6.6). If $J$ belongs to $\mathcal{A}_{\omega_0}^G$, we have that $\hat{Z}^a_j = 0$, for any $a$ in $\mathfrak{g}$. It follows that for any smooth curve $J_t$ in $\mathcal{A}_{\omega_0}^G$, with Chern connection $\nabla_t$, $\mu \hat{z}^a(J_t)$, hence $\int_M f^a s^\nabla \frac{\omega_0}{m!}$, is constant in $t$. □

For any chosen connected component, $\mathcal{V}_{\omega_0}^G$ say, of $\mathcal{A}_{\omega_0}^G$ we then associate the linear map $\mathcal{F}_{\mathcal{V}_{\omega_0}^G} : \mathfrak{g} \to \mathbb{R}$ defined by

$$\mathcal{F}_{\mathcal{V}_{\omega_0}^G}(a) = \int_M f^a s^\nabla \frac{\omega_0}{m!},$$

for any $a$ in $\mathfrak{g}$ and any $J$ in $\mathcal{V}_{\omega_0}^G$, of hermitian scalar curvature $s^\nabla$. By Proposition 9.7.1, the rhs is independent of the choice of $J$ in $\mathcal{U}_{\omega_0}^G$, hence determines a symplectic Futaki invariant relative to $\mathcal{V}_{\omega_0}^G$.

In particular, a necessary condition that the chosen connected component $\mathcal{V}_{\omega_0}^G$ contains an almost-complex structure $J$ whose hermitian scalar curvature is constant is that the symplectic Futaki invariant $\mathcal{F}_{\mathcal{V}_{\omega_0}^G}$ be identically zero on $\mathfrak{g}$.

### 9.8. The Mabuchi K-energy as a Kähler potential

In this section, we show that, under specific conditions, the Mabuchi K-energy can be defined on the space $\mathcal{J}_{\omega_0}$ and interpreted as a Kähler potential. We then infer a new, simple derivation of the scalar curvature as a momentum map for the action of $\text{Ham}_{\omega_0}(M, \omega_0)$ on $\mathcal{J}_{\omega_0}$.

**Proposition 9.8.1.** If the Futaki character $\mathcal{F}_\Omega$ is trivial, the Mabuchi K-energy can be viewed as a $H(M, \omega_0)$-invariant function defined on $\mathcal{J}_{\omega_0}$. If moreover $H^1(M, \mathbb{R}) = \{0\}$, then the Mabuchi K-energy is a Kähler potential of the natural Kähler structure of $\mathcal{J}_{\omega_0}$ defined by (9.2.9)-(9.2.10)-(9.2.11).

**Proof.** Recall that the Mabuchi K-energy $E$ has been defined in Section 4.10 by $\sigma = -dE_\Omega$, where $\sigma$ is the (closed) 1-form on $\mathcal{M}_\Omega$ defined by $\sigma(f) = \int_M f s^\nabla \omega_0$. The 1-form $\sigma$ is $\text{Aut}_{\omega_0}(M, J_0)$-invariant and, by (4.12.2), is basic with respect to the action of $\text{Aut}_{\omega_0}(M, J_0)$ if and only if the Futaki character $\mathcal{F}_\Omega$ is trivial, cf. Remark 4.12.1. It follows that if $\mathcal{F}_\Omega \equiv 0$, $\sigma$ can be viewed as
a 1-form defined on the quotient $\mathcal{M}_{\Omega}/H(M, J_0)$, hence on $J_{\omega_0}/\text{Sp}_0(M, \omega_0)$ via the correspondence (9.1.1). The 1-form $\tilde{\sigma}$ defined by

\begin{equation}
\tilde{\sigma} = p^* \sigma,
\end{equation}

can then be viewed as a basic 1-form on $J_{\omega_0}$ with respect to the action of $\text{Sp}_0(M, \omega_0)$. Similarly, if $\mathcal{F}_\Omega \equiv 0$ the Mabuchi K-energy $E_\Omega$ is constant on each orbit of $H(M, J_0)$-invariant — hence defined on the quotient $\mathcal{M}_{\Omega}/H(M, J_0)$ — and we then define

\begin{equation}
E_\mathcal{J} = p^* E_\Omega,
\end{equation}

as a $\text{Sp}_0(M, \omega_0)$-invariant function on $J_{\omega_0}$, with

\begin{equation}
\tilde{\sigma} = -d E_\mathcal{J}.
\end{equation}

We then set

\begin{equation}
\tau := -J \tilde{\sigma} = d^* E_\mathcal{J}.
\end{equation}

The 1-form $\tau$ defined that way on $J_{\omega_0}$ is still $\text{Sp}_0(M, \omega_0)$-invariant, although no longer basic. It remains to check that

\begin{equation}
d\tau = \kappa_{|J_{\omega_0}}
\end{equation}

— where, we recall, $\kappa$ is the Kähler form defined by (9.2.11) — when $H^1(M, \mathbb{R}) = \{0\}$, i.e. when $\mathfrak{sp} = \mathfrak{ham}$. To compute $d\tau$, it is convenient to use the vector fields induced by the action of $\text{Sp}_0(M, \omega_0)$ on $J_{\omega_0}$, namely those vector fields $\hat{Z}$ given by (9.2.1)-(9.2.2) for any $Z$ in $\mathfrak{sp}$. By (9.1.10) these vector fields $\hat{Z}$, together with the vector fields $J \hat{Z}$, generate the tangent bundle of $J_{\omega_0}$. By (9.1.5), we have that

\begin{equation}
\tau(J \hat{Z}) = -\tilde{\sigma}(\hat{Z}) = 0,
\end{equation}

for any $Z = \text{grad}_{\omega_0} f$ in $\mathfrak{ham}$, whereas $\tau(\hat{Z})$ is given by

\begin{equation}
\tau_J(\hat{Z}) = -\int_M f s_g v_g,
\end{equation}

for any $J$ in $J_{\omega_0}$, where $g = \omega_0(\cdot, J \cdot)$ denotes the riemannian metric determined by the pair $(J, \omega_0)$, $s_g$ the scalar curvature of $g$ and $v_g = \frac{\omega_0^m}{m!}$ the volume form of $g$ (we again emphasize that $v_g$ is actually independent of $J$ in $J_{\omega_0}$). Indeed, if $J = \Phi \cdot J_0$ and if $\tilde{g} = \Phi^* g$ denotes the metric defined by (9.1.7), i.e. the metric determined by the Kähler pair $(J_0, \omega = \Phi^* \omega_0)$, by Proposition 9.1.1 we have that

\begin{equation}
\tau_J(\hat{Z}) = \tilde{\sigma}_J(\hat{Z}) = \sigma(dp(-J L_Z J))
= -\sigma(\Phi^* f)
= -\int_M (\Phi^* f) s_{\tilde{g}} \frac{\omega_0^m}{m!}
= -\int_M f s_g \frac{\omega_0^m}{m!}.
\end{equation}

We now assume that $H^1_{dR}(M, \mathbb{R}) = \{0\}$, i.e. that $\mathfrak{sp} = \mathfrak{ham}$. In order to check (9.8.5), it is then sufficient to compute $d\tau(Z_1, \hat{Z}_2)$ and $d\tau(\hat{Z}_1, J \hat{Z}_2)$, for any two hamiltonian vector fields $Z_1 = \text{grad}_{\omega_0} f_1$ and $Z_2 = \text{grad}_{\omega_0} f_2$. Since $\tau$ is $\text{Sp}_0(M, \omega_0)$-invariant, we have that $d\tau(Z_1, \hat{Z}_2) = \tau([\hat{Z}_1, \hat{Z}_2]) = \tau(J \hat{Z}_1, \hat{Z}_2) = \tau_{\mathcal{J}}(\hat{Z}_1, \hat{Z}_2) = \gamma$.}

\begin{equation}
d\tau = \gamma.
\end{equation}
\[ -\tau([\tilde{Z}_1, \tilde{Z}_2]), \text{ where the bracket } [Z_1, Z_2] \text{ is again a hamiltonian vector field whose momentum is the Poisson bracket } \{f_1, f_2\} = \lambda(df_1 \wedge df_2) = \delta^J(f_1 df_2) \] 

(the first expression is the same as (1.2.10 ; the second one follows by using the identity (1.14.1)). By (9.8.7), we then have

\[
d\tau_{\tilde{J}}(\tilde{Z}_1, \tilde{Z}_2) = \int_M \{f_1, f_2\} s_g v_g \\
= -\langle L_\kappa f_1, f_2 \rangle \\
= 2\langle \delta \delta D^- d^J f_1, f_2 \rangle \\
= 2\langle D^- d^J f_1, D^- df_2 \rangle
\]

by setting \( \kappa = \text{grad}_{\omega_0} s_g = J \text{grad}_g s_g \) and by using (1.23.15) (here, the inner products and the operator \( D^- \) are relative to the Kähler structure \( (g, J, \omega_0) \)). On the other hand, by using (1.23.7) we get

\[
\kappa(\tilde{Z}_1, \tilde{Z}_2) = \frac{1}{2} \langle L_{J Z_1} J, L_{Z_2} J \rangle \\
= 2\langle D^- d^J f_1, D^- df_2 \rangle
\]

which is then equal to \( d\tau_{\tilde{J}}(\tilde{Z}_1, \tilde{Z}_2) \). Now, each \( \tilde{Z} \) preserves \( J \), so that \( [\tilde{Z}_1, J \tilde{Z}_2] = J[\tilde{Z}_1, \tilde{Z}_2] \); we infer

\[
d\tau(\tilde{Z}_1, J \tilde{Z}_2) = -J \tilde{Z}_2 \cdot \tau(\tilde{Z}_1) = J \tilde{Z}_2 \cdot \int_M f_1 s_g v_g.
\]

In the integral, \( f_1 \) and \( v_g = \frac{\omega_0^n}{m!} \) are independent of \( J \), whereas the variation \( \dot{g} \) of the metric \( g = \omega_0(\cdot, J \cdot) \) along \( J \tilde{Z} \) is given by

\[
\dot{g} = \omega(\cdot, J \cdot) = -g(J \dot{J} \cdot, \cdot) = -g(L_{J Z_2} J \cdot, \cdot) = 2D^- df_2.
\]

Now \( \dot{g} \) is \( J \)-invariant, hence traceless and orthogonal to the Ricci tensor of the Kähler structure; from (5.4.4), we then infer \( \dot{s}_g = \delta \delta \dot{g} = 2\delta \delta D^- df_2 \) and we thus get

\[
d\tau(\tilde{Z}_1, J \tilde{Z}_2) = 2\langle f_1, \delta \delta D^- df_2 \rangle = 2\langle D^- df_1, D^- df_2 \rangle
\]

whereas, by using (1.23.7) again we get

\[
\kappa(\tilde{Z}_1, J \tilde{Z}_2) = \frac{1}{2} \langle L_{J Z_1} J, L_{J Z_2} J \rangle = 2\langle D^- df_1, D^- df_2 \rangle = d\tau(\tilde{Z}_1, J \tilde{Z}_2).
\]

By Theorem 9.6.1 we know that the hermitian scalar curvature can be interpreted as a momentum for the natural action of the group \( \text{Ham}_0(M, \omega_0) \) on \( \mathcal{AC}_{\omega_0} \), hence a fortiori for the action of \( \text{Ham}_0(M, \omega_0) \) on \( \mathcal{J}_{\omega_0} \) (on which the hermitian curvature coincides with the riemannian one). By Proposition 9.8.1, we see that the Kähler structure of \( \mathcal{J}_{\omega_0} \) is itself encoded by the scalar curvature, via the Mabuchi K-energy. This observation in turn allows for an alternative, direct interpretation of the scalar curvature as a momentum map for the action of \( \text{Ham}_0(M, \omega_0) \) on \( \mathcal{J}_{\omega_0} \):

**Proposition 9.8.2.** With the above hypotheses, namely \( F_\Omega \equiv 0 \) and \( H^1(M, \mathbb{R}) = \{0\} \), the action of \( \text{Sp}_0(M, \omega_0) \) — which then coincides with
\[ \mu Z(J) = \tau(\hat{Z}(J)) = -\int_M f^Z \frac{\omega^m}{m!}, \]

for any \( Z = \text{grad}\omega_0 f^Z \) in \( \mathfrak{ham} \), where \( s_g \) is the scalar curvature of the metric \( g = \omega_0(\cdot, J\cdot) \) determined by \( J \).

**Proof.** By Proposition 9.8.1, \( \kappa|_{J_0} = d\tau = dd^J\bar{\epsilon} \), where \( \bar{\epsilon} \) and \( \tau = d^J\bar{\epsilon} \) are \( \text{Sp}(M, \omega_0) \)-invariant. For any \( Z \) in \( \mathfrak{ham} \) we then have

\[ 0 = L_Z\tau = d(\tau(\hat{Z})) + \iota_Z d\tau. \]

From this identity — which is in fact quite general for Kähler manifolds whose Kähler form admits an invariant primitive — we infer that \( \tau(\hat{Z}) \), whose value is given by (9.8.7), is a momentum of \( \hat{Z} \) on \( J_0 \).

9.9. The relative case

In this section, we fix a connected compact subgroup \( G \) of the reduced automorphism group \( \text{H}_{\text{red}}(M, J_0) \) and we assume that the background Kähler structure \( (g_0, J_0, \omega_0) \) is \( G \)-invariant. In particular, \( G \) is a subgroup of \( \text{Ham}(M, \omega_0) \), hence of \( \text{Sp}(M, \omega_0) \).

Recall — cf. Section 4.14 — that the identity component of the normalizer, resp. the centralizer, of \( G \) in \( \text{H}(M, J_0) \) as been denoted by \( \text{H}_G(M, J_0) \), resp. \( \text{H}^G(M, J_0) \). Similarly, we denote by \( \text{Diff}_G(M) \), resp. \( \text{Diff}^G(M) \), the identity component of the normalizer, resp. centralizer, of \( G \) in \( \text{Diff}_0(M) \), and by \( \text{Sp}_G(M, \omega_0) \), resp. \( \text{Sp}^G(M, \omega_0) \), the identity component of the normalizer, resp. centralizer, of \( G \) in \( \text{Sp}(M, \omega_0) \). The Lie algebras are denoted accordingly: \( \mathfrak{sp}_G, \mathfrak{sp}^G \), etc. We then have

**Lemma 9.9.1.** Let \( G \) be any connected, compact subgroup of \( \text{H}_{\text{red}}(M, J_0) \) and assume that \( \omega_0 \) is \( G \)-invariant, so that \( G \) is a subgroup of \( \text{Ham}(M, \omega_0) \subset \text{Sp}(M, \omega_0) \). Then

\[ \text{Sp}_G(M, \omega_0)/G = \text{Sp}^G(M, \omega_0)/\hat{Z}(G), \]

by setting \( \hat{Z}(G) = G \cap \text{Sp}^G(M, \omega_0) \).

**Proof.** Since \( \text{Sp}^G(M, \omega_0)/\hat{Z}(G) \) is a subgroup of \( \text{Sp}_G(M, \omega_0) \), and both groups are connected, it is sufficient to check that

\[ \mathfrak{sp}_G = \mathfrak{sp}^G + \mathfrak{g}. \]

The argument is quite similar to the one in the proof of Lemma 4.14.2 and we refer to it for more detail. Let \( Z = Z_H + \text{grad}\omega_0 f^Z \) be any element of \( \mathfrak{sp} \), where \( Z_H \) is the dual of a harmonic 1-form with respect to \( g_0 \). Then, \( Z \) belongs to \( \mathfrak{sp}_G \) if and only if \( [X, Z] = \mathcal{L}_X Z = \mathcal{L}_X Z_H + \text{grad}\omega_0(\mathcal{L}_X f^Z) = \text{grad}\omega_0(\mathcal{L}_X f^Z) \) belongs to \( \mathfrak{g} \) for any \( X \) in \( \mathfrak{g} \). This holds if and only if \( \mathcal{L}_X f^Z \) belongs to the space, \( P^G_{g_0} \), of Killing potentials of elements of \( \mathfrak{g} \) with respect to \( g_0 \). This implies that \( \gamma^* f^Z - f^Z \) belongs to \( P^G_{g_0} \) for any \( \gamma \) in \( G \). By integrating over \( G \), we get \( f^Z = f^F + f \), where \( f^F \) is \( G \)-invariant and \( f \) belongs to \( P^G_{g_0} \). It follows that \( Z = (Z_H + \text{grad}\omega_0 f^Z) + \text{grad}\omega_0 f \) belongs to
1. \( J^G \) as the space of those elements \( J \) in \( J^G \) which satisfy the following two properties:

1. \( J \) is \( G \)-invariant.
2. \( J \) can be connected in \( J^{G}_{\omega_0} \) by a curve \( J_t := \Phi_t \cdot J_0 \), with \( J_1 = J \), where \( \Phi_t \) is a smooth curve in \( \text{Diff}^G(M) \), with \( \Phi_0 = \text{Id} \) (notice that \( \Phi_1 \) is then well-defined up to right composition by an element of \( H^G(M,J_0) \)).

Then, \( J^G_{\omega_0} \) is acted on by \( \text{Sp}_G(M,\omega_0) \) as well as by \( \text{Sp}_G(M,\omega_0) \). By Lemma 9.9.1, these two actions coincide.

**Proposition 9.9.1.** (i) There exists an \( \text{Sp}_G(M,\omega_0) \)-invariant map, \( p^G \), from \( J^G_{\omega_0} \) to the quotient \( M_{\Omega}^G/\text{H}_G(M,J_0) \) and an \( \text{H}_G(M,J_0) \)-invariant map, \( q^G \), from \( M_{\Omega}^G \) to the quotient \( J^G_{\omega_0}/\text{Sp}_G(M,\omega_0) \), which are inverse to each other, hence induce a natural identification

\[
\begin{align*}
\text{identificationbis} & \qquad (9.9.3) \quad J^G_{\omega_0}/\text{Sp}_G(M,\omega_0) \xrightarrow{p^G} M_{\Omega}^G/\text{Aut}_G(M,J_0) \\
\text{propidG-relative} & \qquad (9.9.4) \quad p(J) = \Phi^* \omega_0 = \Phi^{-1} \cdot \omega_0 \mod \text{Aut}_0(M,J_0). \\
\text{identificationbis} & \qquad (9.9.5) \quad q(\omega) = \Phi \cdot J_0 \mod \text{Sp}_0(M,\omega_0).
\end{align*}
\]

(ii) For any \( J \) in \( J_{\omega_0} \) the tangent space \( T_JJ_{\omega_0} \) is described by

\[
T_JJ_{\omega_0} = \{ A = -\mathcal{L}_Z J \mid Z \in \mathfrak{sp} + J\mathfrak{sp} = \mathfrak{sp} \oplus J\mathfrak{ham} \}.
\]

Via this identification, the differential of \( p \) at \( J = \Phi \cdot J_0 \) is given by

\[
dp(-\mathcal{L}_Z J) = \Phi^* (\mathcal{L}_Z \omega_0) = dd^c(\Phi^* f)
\]

for each \( Z = X - JY \in \mathfrak{sp} \oplus J\mathfrak{ham} \), where \( f \) denotes the normalized momentum of the hamiltonian vector field \( Y \), hence is determined by

\[
\mathcal{L}_Z \omega_0 = dd^c f
\]

and \( \int_M f \omega_0^m = 0 \) (here \( d^c \) stands for the operator \( J_0dJ_0^{-1} \), also denoted by \( d^h \), whereas \( d^J := JdJ^{-1} \)).

(iii) \( J_{\omega_0} \) is a complex submanifold, hence a Kähler submanifold of \( \mathcal{C}_{\omega_0} \).

Recall that the relative Futaki character \( \mathcal{F}_{\Omega}^G \) relative to \( G \) and a fixed Kähler class \( \Omega \) has been defined in Section 4.14. Propositions 9.8.1 and 9.8.2 have then the following counterparts:
Proposition 9.9.2. If the relative Futaki character \( F^G_{\Omega} \) is identically zero, the relative K-energy \( E^G_{\Omega} \) can be viewed as a \( \text{Sp}_G(M,\omega_0) \)-invariant function defined on \( J_{\omega_0}^G \). If moreover \( H^1(M,\mathbb{R}) = \{0\} \), then the relative K-energy is a \( \text{Ham}_G(M,\omega_0) \)-invariant Kähler potential of the natural Kähler structure of \( J_{\omega_0}^G \).

Proof. The proof is quite similar to the proof of Proposition 9.8.1; in particular, the condition \( F^G_{\Omega} \equiv 0 \) is equivalent to the fact that \( \tilde{\sigma} \) is basic with respect to the action of \( \text{H}_G(M,J) \), i.e. that the relative Mabuchi K-energy \( E^G_{\Omega} \) is constant along the orbits of \( \text{H}_G(M,J) \). If this condition is satisfied, \( E^G_{\Omega} \) can then be viewed as a \( \text{Sp}_G(M,\omega_0) \)-invariant function, call it \( E^G = \text{J} \tilde{\sigma}^G = d\tilde{\sigma}^G \). We then set \( \tilde{\tau}^G = -\text{J} \tilde{\sigma}^G = d\tilde{\sigma}^G \). It remains to prove that

\[
\kappa_{J_{\omega_0}^G} = d\tilde{\tau}^G = dd^J E^G_{\Omega},
\]

when we assume that \( H^1(M,\mathbb{R}) = 0 \), i.e. that \( \text{Sp}_G(M,\omega_0) = \text{Ham}_G(M,\omega_0) \).

This is checked as in the proof of Proposition 9.8.1, provided we take into account that for any \( X = \text{grad}_{\omega_0} h^X \) in \( \text{ham}_G \), the momentum \( h^X \) is \( G \)-invariant (cf. the observation in the proof of Proposition 4.14.2); this implies that in expressions like \( L_k h^X \), the vector field \( K = \text{grad}_{\omega_0} h \) can be replaced by \( \tilde{K} := \text{grad}_{\omega_0} s_g \); the rest of the computation is identical. □

Proposition 9.9.3. For any maximal compact subgroup \( G \) of \( \text{H}_{\text{red}}(M,J) \) and with the above hypotheses, namely \( F^G_{\Omega} \equiv 0 \) and \( H^1(M,\mathbb{R}) = \{0\} \), the action of \( \text{Sp}_G(M,\omega_0) \) — which then coincides with \( \text{Ham}_G(M,\omega_0) \) — on \( J_{\omega_0} \) is hamiltonian with respect to the natural symplectic form \( \kappa_{J_{\omega_0}^G} \), with momentum \( \mu_G \) given by

\[
\mu^G_{\omega_0}(J) = \tau(\tilde{Z}(J)) = -\int_M f_Z s_g \omega_0^m m!,
\]

for any \( Z = \text{grad}_{\omega_0} h^Z \) in \( \text{ham}_G \), where \( s_g \) denotes the reduced scalar curvature relative to \( G \) of the metric \( g = \omega_0(\cdot,J\cdot) \) determined by \( J \).

Proof. Same argument as for Proposition 9.8.2. □
CHAPTER 10

Extremal Kähler metrics on Hirzebruch-like ruled surfaces

In the same paper [45] where E. Calabi introduced extremal Kähler metrics, he also constructed extremal Kähler metrics — of non constant scalar curvature — on a family of (compact) complex manifolds, which, as it had been known for a long time by using Theorems 3.6.2 and 3.6.1, admit no Kähler metric of constant scalar curvature, a fortiori no Kähler-Einstein metric. These manifolds are the Hirzebruch surfaces, already considered in Section 6.5 and, more generally, the (complex) ruled manifolds of the form $\mathbb{P}(1 \oplus L)$ for any non trivial holomorphic line bundle $L$ over a complex projective space $\mathbb{P}^m$. Here, $\mathbb{P}(1 \oplus L)$ stands for the completion of $L$ obtained by adding an infinity section or, equivalently, the projective line bundle associated to the rank 2 holomorphic vector bundle $1 \oplus L$, where 1 stands for the product complex line bundle (more detail in Section 10.2).

In this chapter, we recall Calabi’s construction for Hirzebruch surfaces — for simplicity, we omit the case when $m > 1$, which is quite similar — and, following [194], [195], we show how Calabi’s method extends to Hirzebruch-like ruled surfaces over Riemann surfaces of all genera, with two main differences however:

1. Hirzebruch surfaces admit a large — non reductive — group of automorphisms, whose maximal compact subgroups — isomorphic to $U(2)/\mu_\ell$ if $L$ is of (negative) degree $-\ell$ — acts with generic orbits of codimension 1. According to Theorem 3.5.1, the issue of constructing extremal metrics can then be reduced a priori to solving an ordinary differential equation. For Hirzebruch-like ruled surfaces of genus greater than 0, the maximal compact subgroups of the group of automorphisms are isomorphic to the circle $S^1$ and Calabi’s method only applies to a subclass of Kähler metrics, which, following [4], [5] we name admissible, cf. Section 10.3.

2. Unlike the genus 0 case — and also the genus 1 case, as observed by A. Hwang in [107] and D. Guan in [96] — Calabi’s method in genera greater than 1 provides (admissible) extremal Kähler metrics on a part only of the Kähler cone, a fact first discovered by C. Tønnesen-Friedman in [194], [195], cf. Theorem 10.4.1.

Following [6], the existence/non-existence issue of — non necessarily admissible — extremal Kähler on Hirzebruch-like ruled surfaces in the part of the Kähler cone which cannot be solved by using Calabi’s approach is eventually settled — cf. Theorem 10.9.1 — by using Chen-Tian Theorem 4.15.2. This requires the explicit computation of the Futaki invariant relative to the natural $\mathbb{C}^*$-action on each Hirzebruch-like rules surface and of the (relative)
K-energy on the space of all (admissible) Kähler metrics in a given Kähler class, which happens to be quite similar to Donaldson’s computation in [77], cf. Sections 10.7, 10.8 and 10.9.

This in turn happens to be a special case of similar Calabi-like constructions and existence/non-existence theorems which appear in [5], [6] in the broader context of admissible projective bundles and admissible Kähler metrics.

10.1. Complex ruled surfaces

A (complex, geometrically) ruled surface of genus \(g\) is a complex surface of the form \(\mathbb{P}(E)\), where \(E\) is a rank 2 complex holomorphic vector bundle over a (connected) compact Riemann surface \(\Sigma\) of genus \(g\) and \(\mathbb{P}(E)\) then denotes the corresponding projectivized bundle, i.e. the holomorphic bundle whose fiber at any point \(y\) of \(\Sigma\) is the complex projective line \(\mathbb{P}(E_y)\), where \(E_y\) denotes the fibre of \(E\) at \(y\).

We denote by \(\mathcal{O}^E(-1)\) the relative tautological line bundle over \(\mathbb{P}(E)\), whose fiber at any point \(x\) of \(\mathbb{P}(E)\), view as a complex line in \(E_y\) for some \(y\) in \(\Sigma\), is \(x\) itself, and by \(\mathcal{O}^E(1)\) the dual of \(\mathcal{O}^E(-1)\).

Notice that \(\mathbb{P}(E)\) remains unchanged if \(E\) is replaced by \(E \otimes L\), for any holomorphic line bundle \(L\), while \(\mathcal{O}^E(-1)\) and \(\mathcal{O}^E(1)\) do depend on \(E\).

In the sequel of this chapter, \(\mathbb{P}(E)\) will be denoted by \(M\), whereas the natural projection from \(M\) to \(\Sigma\) will be denoted by \(\pi\).

We denote by \(\gamma_E\) the first Chern class of \(\mathcal{O}^E(1)\) in \(H^2(M, \mathbb{Z})\) and by \(\gamma_\Sigma\) the (positive) generator of \(H^2(\Sigma, \mathbb{Z}) = \mathbb{Z}\) (here, positive refers to the orientation of \(\Sigma\) determined by its complex structure). It is a simple consequence of the Leray-Hirsch theorem — cf. e.g. [36] — that \(H^*(M, \mathbb{Z})\) is a free module over \(H^*(\Sigma, \mathbb{Z})\) generated by \(1\) and \(\gamma_E\). In particular, \(H^2(M, \mathbb{Z})\) has no torsion, hence can be considered as a lattice in \(H^2(M, \mathbb{R})\), and the pair \(\gamma_E, \pi^*\gamma_\Sigma\) is then a \(\mathbb{R}\)-basis of \(H^2(M, \mathbb{R})\).

Moreover, both \(\gamma_E\) and \(\pi^*\gamma_\Sigma\) are of type \((1, 1)\): this is because \(\gamma_E\) is the Chern class of a holomorphic line bundle over \(M\), whereas \(H^2(\Sigma, \mathbb{R})\) coincides with \(H^{1,1}(\Sigma, \mathbb{R})\) as \(\Sigma\) is of complex dimension 1. We then have \(H^2(M, \mathbb{R}) = H^{1,1}(M, \mathbb{R})\).

The ring structure of \(H^*(M, \mathbb{Z})\) is determined by the ring structure of \(H^*(\Sigma, \mathbb{Z})\) and the relation

\[
\gamma_E \cdot \gamma_E + \gamma_E \cdot \pi^*c_1(E) = 0,
\]

where \(c_1(E) = \deg(E)\gamma_\Sigma\) is the first Chern class of \(E\), cf. [36]).

The table for the cup product — or intersection form — in \(H^2(M, \mathbb{Z})\), with values in \(H^4(M, \mathbb{Z}) = \mathbb{Z}\) is then

\[
\gamma_E \cdot \gamma_E = -\deg(E), \quad \gamma_E \cdot \pi^*\gamma_\Sigma = +1, \quad \pi^*\gamma_\Sigma \cdot \pi^*\gamma_\Sigma = 0;
\]

the third identity is clear, as \(H^4(\Sigma, \mathbb{Z}) = \{0\}\); the second one follows from \(\gamma_\Sigma\) being Poincaré dual of a point in \(H_0(\Sigma, \mathbb{Z})\) — so that \(\pi^*\gamma_\Sigma\) is Poincaré dual of the element of \(H_2(M, \mathbb{Z})\), say \(F\), represented by any fiber of \(\pi\) — whereas \(\int_F \gamma_E = 1\), as the restriction of \(\gamma_E\) to any fiber \(\pi^{-1}(x)\) is the Chern class of the dual tautological line bundle over \(\pi^{-1}(x)\), which is of degree 1; the first identity then follows from (10.1.1).
It is convenient to substitute to $\gamma_E$ — which depends upon the choice of $E$ in the expression of $M = \mathbb{P}(E)$ — the element $\epsilon$ of $H^2(M, \mathbb{Q})$ defined by

$$\epsilon := \gamma_E + \frac{\deg E}{2} \pi^* \gamma_{\Sigma},$$

which only depends on $M$. As a matter of fact, $\epsilon$ is the Chern class of the positive square root of the vertical tangent bundle on $\mathbb{P}(E)$ cf. [83]. Indeed, since $c_1(E) = c_1(\Lambda^2 E)$, $2\epsilon$ is the Chern class of the holomorphic line bundle $\mathcal{O}^E(2) \otimes \pi^* \Lambda^2 E$, whose fiber at any point $x$ of $M$ over $y$ — viewed as a complex line in $E_y$ — is the tensor product $x^* \otimes x^* \otimes \Lambda^2 E_y$; this is canonically identified with $x^* \otimes (E_y/x)$, hence with $T_x(\mathbb{P}(E_y))$, cf. Remark 6.2.5.

For any Riemann surface $\Sigma$, the first Chern class of the (holomorphic) tangent bundle of $\Sigma$ is $\chi(\Sigma) \gamma_{\Sigma} = 2(1 - g) \gamma_{\Sigma}$; from the foregoing, we then infer that the first Chern class of $M$ in $H^2(M, \mathbb{Z})$ is given by

$$c_1(M) = 2\epsilon + 2(1 - g) \pi^* \gamma_{\Sigma}.$$

The pair $\epsilon, \pi^* \gamma_{\Sigma}$ provides an alternative basis of $H^2(M, \mathbb{R})$, with the following table for the product:

<table>
<thead>
<tr>
<th>Product</th>
<th>$\epsilon \cdot \epsilon = 0$</th>
<th>$\epsilon \cdot \pi^* \gamma_{\Sigma} = 1$</th>
<th>$\pi^* \gamma_{\Sigma} \cdot \pi^* \gamma_{\Sigma} = 0$</th>
</tr>
</thead>
</table>

Following [83], define $t(E)$ by

$$t(E) = \max(\deg(F)) - \frac{\deg(E)}{2},$$

where $F$ runs over the space of all proper holomorphic subline bundles of $E$.

Then, $E$ is said to be stable if $t(E) < 0$; semi-stable if $t(E) \leq 0$; unstable if $t(E) > 0$; polystable — or quasistable — if $E$ is either stable or can be written as $E = L_1 \oplus L_2$, where $L_1, L_2$ are holomorphic line bundles over $\Sigma$ of the same degree. In general, if $E$ is decomposable, i.e. of the form $E = L_1 \oplus L_2$ for any two holomorphic line bundles, with $\deg(L_1) \geq \deg(L_2)$, we easily check that

$$t(E) = \frac{1}{2}(\deg(L_1) - \deg(L_2));$$

$E$ is then unstable, unless it is polystable. Notice that $t(E)$ is unchanged when $E$ is replaced by $E \otimes L$; the above conditions then qualify the ruled complex surface $M = \mathbb{P}(E)$.

A celebrated theorem of M. S. Narasimhan and C. S. Seshadri [153] asserts that $E$ is polystable if and only if $\mathbb{P}(E)$ arises from an representation, say $\rho : \Gamma \to PU(2)$, of the fundamental group, $\Gamma = \pi_1(\Sigma)$, of $\Sigma$ to the projectivized unitary group $PU(2)$, meaning that $M = \mathbb{P}(E)$ can be realized as

$$M = (\hat{\Sigma} \times \mathbb{P}^1) / \Gamma,$$

where $\hat{\Sigma}$ denotes the universal covering of $\Sigma$ and $\Gamma$ acts on the product $\hat{\Sigma} \times \mathbb{P}^1$ via its covering action on $\hat{\Sigma}$ and via the representation $\rho$ on $\mathbb{P}^1$. Recall — cf. Proposition 6.2.2 — that $PU(2)$ is the identity component of the isometry group of the complex projective line $\mathbb{P}^1$ equipped with a standard Fubini-Study metric. It follows that the action of $\Gamma$ on the product $\hat{\Sigma} \times \mathbb{P}^1$ is isometric whenever $\mathbb{P}^1$ is equipped with a standard Fubini-Study metric — of
any (positive) constant sectional curvature — and \( \Sigma \) with a metric arising from a Kähler metric of constant sectional curvature on \( \Sigma \), equal to a positive multiple of the Euler characteristic \( \chi(\Sigma) = 2(1 - g) \) of \( \Sigma \). Denote by \( \omega_{FS} \) the Kähler form of the standard Fubini-Study of \( \mathbb{P}^1 \) of constant sectional curvature +2 and by \( \tilde{\omega}_\Sigma \) the pull back on \( \tilde{\Sigma} \) of the Kähler form of a Kähler metric on \( \Sigma \) of constant sectional curvature equal to \( \chi(\Sigma) \); by the Gauss-Bonnet formula \(^{1}\), the volume of \( (\mathbb{P}^1, \omega_{FS}) \) and the volume of \( (\Sigma, \omega_\Sigma) \) are both equal to \( 2\pi \). For any two positive real numbers \( A, B \), \( A\omega_{FS} + B\omega_\Sigma \) is then the pull-back of the Kähler form of a Kähler metric on \( M \), whose Kähler class in \( H^2(M, \mathbb{R}) \) is equal to \( \Omega_{(A,B)} = 2\pi(A\epsilon + B\pi^*\gamma_\Sigma) \), and whose (constant) scalar curvature is equal to \( s = \frac{4}{A} + \frac{4(1 - g)}{B} \).

The Kähler cone of \( M \) is contained in the positive cone generated by \( \epsilon \) and \( \pi^*\gamma_\Sigma \), i.e. the set of elements \( \Omega \) of \( H^2(M, \mathbb{R}) \) defined by (10.1.10) for any two positive numbers: indeed, if \( \Omega \) is a Kähler class, \( \int_M \Omega = 2\pi A \int_M \gamma_E = 2\pi A \) must be positive, as well as \( \int_M \Omega^2 \), which, by (10.1.5), is equal to \( 8\pi^2 AB \), cf. [83]. When \( M \) is polystable, it readily follows from the above that the Kähler cone is then exactly the positive cone generated by \( \epsilon \) and \( \pi^*\gamma_\Sigma \) and that any Kähler class \( \Omega_{(A,B)} \) contains the Kähler form of a metric which is locally the product of two Riemann surfaces of constant sectional curvature, equal to \( \frac{2}{A} \) and \( \frac{2(1 - g)}{B} \); in particular, this metric is of constant scalar curvature given by (10.1.11).

For a general complex ruled surface \( M = \mathbb{P}(E) \), the Kähler cone of \( M \) is given by

\[ \text{Theorem 10.1.1 (A. Fujiki [83] Proposition 1). The Kähler cone of a complex ruled surface } M = \mathbb{P}(E) \text{ is the cone of elements of } H^2(M, \mathbb{R}) \text{ given by (10.1.10) for any two positive real numbers } A, B \text{ satisfying the additional condition:} \]

\[ \frac{B}{A} > \max(0, t(E)), \]

where \( t(E) \) is defined by (10.1.6).

For the proof, we refer the reader to Fujiki’s original paper [83].

10.2. Hirzebruch-like ruled surfaces

The ruled complex surfaces of main interest in this chapter are those \( M = \mathbb{P}(E) \) such that \( E \) is unstable and decomposable, i.e. \( E = L_1 \oplus L_2 \), with \( \deg(L_1) > \deg(L_2) \). By substituting \( E \otimes L_1^* \), we can assume that \( E = 1 \oplus L \), where, we recall, \( 1 \) stands for the product bundle \( M \times \mathbb{C} \) and \( L \)

\[ \int_M k_g v_g = 2\pi \chi(\Sigma) = 4\pi (1 - g), \] for any riemannian metric \( g \), of (sectional) curvature \( k_g \), cf. e.g. [99].
$L$ denotes a holomorphic line bundle of \textit{negative} degree $\deg(L) = -\ell$, with $\ell > 0$.

If $g = 0$, i.e. if $\Sigma \cong \mathbb{P}^1$, these are the Hirzebruch surfaces $\mathbb{F}_\ell$ already considered in Section 6.5. In particular, as a complex surface, $\mathbb{F}_1$ is isomorphic to the blow-up of the complex projective plane $\mathbb{P}^2$ along one point. Moreover, except for the product $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}(1 \oplus 1)$, all complex ruled over $\mathbb{P}^1$ are of this type, i.e. are Hirzebruch surfaces: this is because any holomorphic vector bundle over the complex projective line is decomposable — cf. e.g. [24, Proposition III.15] — and any holomorphic line bundle of degree $d$ is isomorphic to $\mathcal{O}(d)$.

In general, a ruled surface $M$ of the above kind will be called a \textit{Hirzebruch-like ruled surface} of genus $g$ and of degree $\ell$, meaning that $M = \mathbb{P}(1 \oplus L)$, where $L$ is a holomorphic line bundle of negative degree $-\ell$ over a (compact) Riemann surface $\Sigma$ of genus $g$ \footnote{When $g > 1$, these are the \textit{pseudo-Hirzebruch surfaces} studied by C. Tønnesen-Friedman in [194], [195].}.

Any Hirzebruch-like ruled surface $M = \mathbb{P}(1 \oplus L)$ can be regarded as the compactification of $L$ obtained by adding a point at infinity to each fiber $L_y$, namely the complex line $L_y$ itself, viewed as a point in $\mathbb{P}(E_y) = \mathbb{P}(\mathbb{C} \oplus L_y)$, cf. Remark 6.5.1. The union of these points at infinity is the \textit{infinity section}, denoted by $\Sigma_\infty$, whereas the zero section of $L$, viewed as a (holomorphic) section of $\mathbb{P}(1 \oplus L)$ via the inclusion $L \subset \mathbb{P}(1 \oplus L)$, is denoted by $\Sigma_0$ (here and henceforth we identify the zero section $\sigma_0 : y \mapsto \mathbb{C}$ and the infinity section $\sigma_\infty : y \mapsto L_y$ of $\mathbb{P}(E)$ with their images, $\Sigma_0$ and $\Sigma_\infty$, in $M = \mathbb{P}(1 \oplus L)$: $\Sigma_0$, resp. $\Sigma_\infty$, is then a copy of $\Sigma$ and its normal bundle in $M$ is holomorphically isomorphic to the restriction of $\pi^*(L)$, resp. $\pi^*(L^*)$).

Denote by $M_0$ the open set of $M$ obtained by removing the zero section $\Sigma_0$ and the infinity section $\Sigma_\infty$; we then have

\begin{equation}
M = \Sigma_0 \cup M_0 \cup \Sigma_\infty.
\end{equation}

\textbf{Remark 10.2.1.} The open set $M_0$ is equivalently defined as $M_0 = L \setminus \Sigma_0$, i.e. the set of non-zero elements of $L$, and can then be regarded as the total space of the $\mathbb{C}^*$-principal bundle of “frames” of $L$. As a complex manifold, $M = \mathbb{P}(1 \oplus L)$ itself can then viewed as the associated bundle\footnote{In general, for any $G$-principal bundle $P$ over some manifold $N$, and for any (smooth) $G$-action on some manifold $F$, the \textit{associated bundle} $M = P \times_G F$ is defined as the quotient $P \times F/G$ for the (right) $G$-action on $P \times F$ defined by $\gamma \cdot (s, u) = (s \gamma, \gamma^{-1} \cdot u)$, for any $\gamma$ in $G$, any $s$ in $P$, any $u$ in $F$. If $P$ is equipped with a connection, determined either by a $G$-equivariant $g$-valued connection 1-form $\eta$ or by the corresponding $G$-invariant horizontal distribution $H = \ker \eta$, then, for any $G$-space $F$, the associated bundle $P \times_G F$ comes naturally equipped with a connection, whose horizontal distribution is induced by the $G$-invariant distribution, say $H$ again, on $P \times F$ defined as follows: for any pair $(s, u)$ in $P \times F$, $H(s, u) := H_s \otimes \{0\}$.} $M_0 \times_{\mathbb{C}^*} \mathbb{P}^2$ determined by the diagonal $\mathbb{C}^*$-action: $\zeta \cdot (z_1, z_2) = (\zeta z_1, \zeta z_2)$ on $\mathbb{C}^2$ and the induced action on $\mathbb{P}^1$). If $Q$ denotes the set of elements of norm 1 with respect to $h$ in $L$, viewed as a $S^1$-principal bundle over $\Sigma$, $M$ can be likewise regarded as the associated bundle $Q \times_{S^1} \mathbb{P}^1$ with respect to the standard $S^1$-action on $\mathbb{P}^1 = S^2$. 

\section{10.2. Hirzebruch-like ruled surfaces}

\textbf{Mdec}
Elements of $H^2(M, \mathbb{Z})$, in particular the generators $\gamma_E$ and $\pi^*\gamma_\Sigma$, will be conveniently represented by their Poincaré duals in $H^2(M, \mathbb{Z})$: $\pi^*\gamma_\Sigma$ is thus identified with the class, $F$, of any fiber of $\pi$ (see Section 10.1), whereas $\gamma_E$ can be represented by the (homology class of the) infinity section $\Sigma_\infty$: this is because the (holomorphic) projection of $E = 1 \oplus L$ to the trivial line bundle $1 = \Sigma \times \mathbb{C}$ along $L$ determines a holomorphic section of $\mathcal{O}^E(1)$, whose zero divisor is $\Sigma_\infty$: it follows that $\Sigma_\infty$ is Poincaré dual of $c_1(\mathcal{O}^E(1)) = \gamma_E$, cf. footnote of page 154. Similarly, the (holomorphic) projection of $E = 1 \oplus L$ to $L$ along the trivial line bundle $1 = \Sigma \times \mathbb{C}$ determines a holomorphic section of $\mathcal{O}^E(1) \otimes \pi^*L$, whose zero divisor is $\Sigma_0$: $\Sigma_0$ is then the Poincaré dual of $c_1(\mathcal{O}^E(1) \otimes \pi^*L) = c_1(\mathcal{O}^E(1)) + \pi^*c_1(L)$; since $c_1(L) = -\ell \gamma_\Sigma$, we eventually get:

$$
(10.2.2) \quad \Sigma_0 = \Sigma_\infty - \ell F
$$

and the table (10.1.2) then becomes

$$
(10.2.3) \quad \begin{align*}
\Sigma_\infty \cdot \Sigma_\infty &= \ell, \\
\Sigma_\infty \cdot F &= 1, \\
F \cdot F &= 0,
\end{align*}
$$

The class $\epsilon$ defined in Section 10.1 can then be written as

$$
(10.2.4) \quad \epsilon = \frac{1}{2}(\Sigma_0 + \Sigma_\infty).
$$

Since $\ell > 0$, (the Poincaré duals of) $\Sigma_0$ and $\Sigma_\infty$ form a basis of $H^2(M, \mathbb{R})$ (but not a set of generators of $H^2(M, \mathbb{Z})$, unless $\ell = 1$). We then have:

**Proposition 10.2.1.** For any Hirzebruch-like ruled surface $M = \mathbb{P}(E)$, the Kähler cone of $M$ is the set of classes $\Omega$ in $H^2(M, \mathbb{R})$ of the form

$$
(10.2.5) \quad \Omega_{a,b} = 2\pi(-a\Sigma_0 + b\Sigma_\infty),
$$

for any two real numbers $a, b$ satisfying the condition

$$
(10.2.6) \quad 0 < a < b.
$$

Moreover, $\frac{\Omega_{a,b}}{2\pi}$ belongs to $H^2(M, \mathbb{Z})$ if and only if $a, b$ both belong to $\mathbb{Z}[\frac{1}{\ell}]$ and the difference $b - a$ is a (positive) integer; we then have

$$
(10.2.7) \quad \frac{\Omega_{a,b}}{2\pi} = (b - a) \gamma_E + a\ell \pi^*\gamma_\Sigma,
$$

meaning that $\frac{\Omega_{a,b}}{2\pi}$ is the Chern class of the holomorphic line bundle $\mathcal{O}^E(b - a) \otimes a\ell \pi^*(L^{-a})$.

**Proof.** For $E = 1 \oplus L$, with $\deg(L) = -\ell < 0$, we readily infer from (10.1.7) that $t(E) = \ell \frac{1}{2}$, whereas, from the above tables we easily deduce that the real numbers $a, b$ in (10.2.5) are related to the real numbers $A, B$ in (10.1.10) by: $a = \frac{A}{2} - \frac{\ell}{2}, \quad b = \frac{B}{2} + \frac{\ell}{2}$; Proposition 10.2.1 is then a direct consequence of Theorem 10.1.1. We here provide a more specific argument. We first show that the condition $0 < a < b$ is necessary by using the general fact that the integral of a Kähler form on any (compact) complex submanifold of positive dimension $p$ is equal to its riemannian volume multiplied by $p!$, hence is positive. By integrating $\Omega$ given by (10.2.5) over $F$, $\Sigma_0$, $\Sigma_\infty$ and assuming that $\Omega$ belongs to the Kähler cone, we get $\int_F \Omega = 2\pi(b - a) > 0$.
The group of automorphisms of complex ruled surfaces has been studied by M. Maruyama in [147], from which we extract the following pieces of information for $H(M)$ — the identity component of the automorphism group of $M$ — in the special case when $M = \mathbb{P}(1 \oplus L)$ is a Hirzebruch-like ruled surface of genus $g$ and degree $\ell$ (cf. also [194] for the case when of Hirzebruch-like ruled surfaces of genus greater than 1).

We first observe that the natural action of $\mathbb{C}^*$ on $L$ extends to a (holomorphic) $\mathbb{C}^*$-action on $M = \mathbb{P}(1 \oplus L)$ as follows: any point $y$ of $M$ in $\pi^{-1}(x)$ is a complex line of $\mathbb{C} \oplus L_x$ generated by, say, the pair $(z, u)$, where $z$ belongs to $\mathbb{C}$ and $u$ belongs to $L_x$; for any $\zeta$ in $\mathbb{C}^*$, $\zeta \cdot y$ is then the complex line of $\mathbb{C} \oplus L_x$ generated by the pair $(z, \zeta u)$. This action clearly preserves the fibration $\pi$ from $M$ to $\Sigma$, pointwise fixes $\Sigma_0$ and $\Sigma_\infty$ and is free — as well as the induced action of $S^1 \subset \mathbb{C}^*$ — on the open set $M_0 = L \setminus \Sigma_0$.

In the sequel of this chapter, the generator of the natural $S^1$-action will be denoted by $T$.

Denote by $H^0(\Sigma, L^*)$ the complex vector space of holomorphic sections of the dual bundle $L^*$. Then, $H^0(\Sigma, L^*)$ can be viewed as a subgroup of $H(M)$. For any element, $\sigma$, of $H^0(\Sigma, L^*)$, the corresponding automorphism of $M$, say $\tilde{\sigma}$, is defined as follows: for any $y$ in $\pi^{-1}(x)$, generated by the pair $(z, u)$ as above, $\tilde{\sigma} \cdot y$ is defined as the complex line of $\mathbb{C} \oplus L_x$ generated by the pair $(z + \sigma(u), u)$. This action preserve each fiber of $\pi$ and pointwise preserves $\Sigma_0$, but does not preserve $\Sigma_\infty$. Notice that the inverse of $\tilde{\sigma}$ is $\check{\sigma}$ and that $\zeta \circ \tilde{\sigma} \circ \zeta^{-1} = \zeta^{-1} \check{\sigma}$.

By putting these two actions together, we get a subgroup of $H(M)$ which is the semi-direct product $\mathbb{C}^* \ltimes H^0(\Sigma, L^*)$ for the linear action $\zeta \cdot \sigma = \zeta^{-1} \check{\sigma}$ of $\mathbb{C}^*$ on $H^0(\Sigma, L^*)$ (cf. footnote 3 of page 162 for the definition of semi-direct products).

Denote by $H^\Sigma(M)$ the subgroup of $H(M)$ whose elements, say $\gamma$, satisfy $\pi \circ \gamma = \pi$. Then, $H(M)$ is described by the following proposition:

**Proposition 10.2.2.** For any Hirzebruch-like ruled surface $M = \mathbb{P}(1 \oplus L)$ of any genus $g$ and any (positive) degree $\ell$, we have

\[
H^\Sigma(M) = \mathbb{C}^* \ltimes H^0(\Sigma, L^*).
\]

Moreover:
(i) If \( g > 0 \), we have
\[
H(M) = H^\Sigma(M) = \mathbb{C}^* \times H^0(\Sigma, L^*).
\]
(ii) If \( g = 0 \), i.e. if \( M = \mathbb{F}_\ell \) for some \( \ell \), we have the following exact sequence:
\[
1 \to \mathbb{C}^* \times H^0(\mathbb{P}^1, \mathcal{O}(\ell)) \to H(\mathbb{F}_\ell) \to PGL(2, \mathbb{C}) \to 1
\]
and the isomorphism
\[
H(\mathbb{F}_\ell) = GL(2, \mathbb{C})/\mu_\ell \times H^0(\mathbb{P}^1, \mathcal{O}(\ell)),
\]
for the action of \( GL(2, \mathbb{C})/\mu_\ell \) on \( H^0(\mathbb{P}^1, \mathcal{O}(\ell)) \) induced by the natural action of \( GL(2, \mathbb{C}) \) on \( \mathcal{O}(-1) \). In this isomorphism, \( GL(2, \mathbb{C})/\mu_\ell \) is the subgroup of elements of \( H(\mathbb{F}_\ell) \) which (globally) preserve \( \Sigma_0 \) — as do all elements of \( H(M) \) — and \( \Sigma_\infty \).

In all cases, we have that
\[
H(M) = H_{\text{red}}(M),
\]
where, we recall, \( H_{\text{red}}(M) \) denotes the reduced automorphism group defined in Section 2.4.

**Proof.** The first assertion follows from [147], Theorem 1, Theorem 2, Lemma 6. Then, (i) follows from [147] Lemma 7 and the exact sequence (10.2.11) in (ii) from [147] Lemma 8 (recall that \( PGL(2, \mathbb{C}) \) is the group of automorphisms of \( \mathbb{P}^1 \) — cf. Proposition 6.1.2 — whereas \( H^0(\mathbb{P}^1, \mathcal{O}(\ell)) = \mathcal{O}(\ell)(\mathbb{C}^2)^* \), which is of (complex) dimension \( \ell + 1 \), cf. Proposition 6.1.1). The linear group \( GL(2, \mathbb{C}) \) acts on \( \mathcal{O}(-1) \) via the natural isomorphism \( \mathcal{O}(-1) \setminus \Sigma_0 = \mathbb{C}^2 \setminus \{0\} \); this action covers the natural action of \( PGL(2, \mathbb{C}) = GL(2, \mathbb{C})/\mathbb{C}^* \) on \( \mathbb{P}^1 \) readily extends to \( F_1 = \mathbb{P}(1 \oplus \mathcal{O}(-1)) \); moreover, the (holomorphic, linear) action of \( GL(2, \mathbb{C}) \) on \( \mathcal{O}(-1) \) induces a (holomorphic, linear) action on \( L^* = \mathcal{O}(1) \), hence on \( H^0(\mathbb{P}^1, \mathcal{O}(1)) \), which extends the above action of \( \mathbb{C}^* \); the semi-direct product \( GL(2, \mathbb{C}) \times H^0(\mathbb{P}^1, \mathcal{O}(1)) \) is then contained in \( H(\mathbb{F}_\ell) \) and fits into the same exact sequence (10.2.11); it follows that \( GL(2, \mathbb{C}) \times H^0(\mathbb{P}^1, \mathcal{O}(1)) \) coincides with \( H(\mathbb{F}_\ell) \) (this isomorphism had already be obtained in Section 6.4, via the identification of \( F_1 \) with the blow-up of \( \mathbb{P}^2 \) along a point). Now, the above action of \( GL(2, \mathbb{C}) \) on \( \mathcal{O}(-\ell) \) induces an action on \( \mathcal{O}(-\ell) \) which is effective on the quotient \( GL(2, \mathbb{C}) \); the same reasoning then gives (10.2.12). All elements of \( H(\mathbb{F}_\ell) \) globally preserve \( \Sigma_0 \), as \( \Sigma_0 \) is the only complex curve of negative self-intersection (this also follows from the isomorphism (10.2.12), since all elements of \( GL(2, \mathbb{C}) \) and \( H^0(\mathbb{P}^1, \mathcal{O}(\ell)) \) preserve \( \Sigma_0 \); on the other hand, all elements of \( GL(2, \mathbb{C}) \) preserve \( \Sigma_\infty \), whereas no element of \( H^0(\mathbb{P}^1, \mathcal{O}(\ell)) \) do preserve \( \Sigma_\infty \), except 0. The last assertion (10.2.13) is clear as the Jacobi map from \( M \) to \( Alb(M) \) — see Section 2.4 — factors through \( \pi \): if \( g = 0 \), the Albanese torus is trivial, whereas, if \( g > 0 \), the Jacobi map is trivial because of (10.2.10). □

**Remark 10.2.2.** If \( M = \mathbb{F}_\ell \), the open set \( M_0 \) appearing in (10.2.1) is biholomorphically isomorphic to \( (\mathbb{C}^2 \setminus \{0\})/\mu_\ell \): it is preserved by the \( GL(2, \mathbb{C})/\mu_\ell \) part of \( H(\mathbb{F}_\ell) \), whose action is then the natural action of \( GL(2, \mathbb{C})/\mu_\ell \) on \( (\mathbb{C}^2 \setminus \{0\})/\mu_\ell \).

As an direct consequence of Proposition 10.2.2 we get
Proposition 10.2.3. For any Hirzebruch-like ruled surface \( M \) of genus \( g \) and degree \( \ell \), the maximal compact subgroups of \( H(M) = H_{\text{red}}(M) \) are conjugate to

(i) \( S^1 \), if \( g > 0 \);
(ii) \( U(2)/\mu_\ell \), if \( g = 0 \).

Proof. In any semi-direct product of the form \( G \ltimes V \), where \( V \) is a vector space and \( G \) is a (connected) Lie group acting linearly on \( V \) — cf. footnote 3 of page 162 — the maximal compact subgroups of \( G \ltimes V \) are clearly the maximal compact subgroups of \( G \). Proposition 10.2.3 then readily follows from Proposition 10.2.2. \( \square \)

10.3. Admissible Kähler metrics

We now equip \( L \) with a hermitian inner product, \( h \), in such a way that the opposite of the curvature form of the corresponding Chern connection be the Kähler form, say \( \omega_L \), of a metric of constant curvature \( \kappa \). Since \( \int_M \omega_L = 2\pi \ell \), it follows from the Gauss-Bonnet formula (10.1.9) that

\[
\kappa = \frac{\chi(\Sigma)}{\ell} = \frac{2(1-g)}{\ell}.
\]

The norm function determined on \( L \) by \( h \) is denoted by \( r \); we then have \( r(u) = (h(u,u))^{\frac{1}{2}} \) for any \( u \) in \( L \). On \( M_0 \), \( r \) is positive and we then set:

\[
t = \log r.
\]

The 1-form \( d^c t \) — or, rather, its restriction to the space, \( Q \), of unit elements of \((L,h)\), viewed as a \( S^1 \)-principal bundle over \( \Sigma \) — is then the connection 1-form of the Chern connection determined by \( h \); on \( M_0 \), we then have:

\[
d^c t(T) = 1,
\]

where, we recall, \( T \) denotes the generator of the action of \( S^1 \) on \( M \) induced by the natural \( S^1 \)-action on \( L \), cf. Section 10.2, whereas

\[
d d^c t = \pi^* \omega_L,
\]

cf. (1.7.3). Moreover, \( e^{2t} dt \wedge d^c t = dr \wedge d^c r \) restricts to the volume form determined by \( h \) on each fiber of \( L \setminus \Sigma_0 \) (here and henceforth, the twisted differential \( d^c \) is relative to the natural complex structure of \( M \), which will be denoted by \( J \)). In particular, \( dd^c t \) smoothly extends to \( M \), \( e^{2t} dt \wedge d^c t \) smoothly extends to \( \Sigma_0 \) and \( e^{-2t} dt \wedge d^c t \) smoothly extends to \( \Sigma_\infty \).

When \( g = 0 \), i.e. when \( M = \mathbb{P}_\ell \) is a (genuine) Hirzebruch surface, all maximal compact subgroup of \( H(M) = H_{\text{red}}(M) \) are conjugate to \( U(2)/\mu_\ell \), cf. Proposition 10.2.3, which acts transitively on \( \Sigma_0 \) and \( \Sigma_\infty \) and acts with cohomogeneity 1 in \( M_0 \), with orbits the distance spheres \( t = \text{constant} \), all isomorphic to \( S^3/\mu_\ell \). By Theorem 3.5.1, any extremal Kähler metric on \( \mathbb{P}_\ell \) — if any — can be made \( U(2)/\mu_\ell \)-invariant by the action of an element of \( H(M) \). We then have:

Proposition 10.3.1. Any \( U(2)/\mu_\ell \)-invariant Kähler metric defined on \( M_0 = (\mathbb{C}^2 \setminus \{0\})/\mu_\ell \) admits a \( U(2)/\mu_\ell \)-invariant globally defined Kähler potential, unique up to an additive constant.
proof. Without loss of generality, we can assume $\ell = 1$, as $\mathbb{C}^2 \setminus \{0\}$ is a covering of $(\mathbb{C}^2 \setminus \{0\})/\mu_\ell$. Let $\omega$ be the Kähler form of a $U(2)$-invariant Kähler metric defined on $M_0 = \mathbb{C}^2 \setminus \{0\}$. Since $M_0$ is simply-connected, we have $\omega = d\alpha$, where $\alpha$ is a real 1-form, which can be chosen $U(2)$-invariant as well (if not, replace $\alpha$ by $\tilde{\alpha} := \int_{U(2)} \gamma^* \alpha \, d\gamma$, where $\gamma$ stands for any bi-invariant measure on the compact Lie group $U(2)$). Since $\omega$ is of type $(1,1)$, we have that $\partial \alpha^{0,1} = 0$; it then remains to show that $\alpha^{0,1} = \bar{\partial} F$, for some $U(2)$-invariant complex function; if so, $\operatorname{Im} F$ would be the required $U(2)$-invariant Kähler potential. On $\mathcal{U}$, the $U(2)$-orbits are the distance spheres of radius $r$, $r > 0$, for the standard euclidean structure of $\mathbb{C}^2$. For each $S_r$, denote by $V_r$ the $U(2)$-invariant tubular neighbourhood of $S_r$ defined by $V_r = \{ u \in \mathcal{U} \mid \frac{r^2}{2} < |u|^2 < \frac{3}{2}r^2 \}$. Then, $V_r$ is contained in the polydisk $D_{2r} \times D_{2r} \subset \mathbb{C}^2$, which is a Stein manifold. It follows that $\alpha^{0,1}$ is $\bar{\partial}$-exact on $D_{2r} \times D_{2r}$, hence, a fortiori, on $V_r$; for each $r$, we then get $F_r$, which can be chosen $U(2)$-invariant, such that $\alpha^{0,1}_{|V_r} = \bar{\partial} F_r$. Moreover, since $U(2)$-invariant holomorphic functions are necessarily constant (easy verification), the the $F_r$’s differ by a constant. Since the $F_r$’s are entirely determined by their restrictions on any radial half-line, ray $D = \mathbb{R}^{>0}$, we eventually get on open covering of $\mathbb{R}^{>0}$, parametrized by $r$, and, on each open interval $I_r = (\frac{\sqrt{3}}{2}r, \frac{\sqrt{3}}{2}r)$, a complex function, still denoted $F_r$, in such a way that $F_r - F_{r'}$ is constant for any $r, r'$ such that $I_r \cap I_{r'} \neq \emptyset$. Since, $H^1(\mathbb{R}^{>0}, \mathbb{C}) = \{0\}$, we infer that there exists $F$ globally defined on $\mathbb{R}^{>0}$ such that $F_r = F_{|V_r}$ on each $V_r$; this, $F$ can be be viewed as a $U(2)$-invariant function defined on $\mathcal{U}$ and we then have $\alpha^{0,1} = \bar{\partial} F$. If $\varphi, \varphi'$ are two $U(2)$-invariant Kähler potential defined on $\mathcal{U}$, then $\varphi' - \varphi$ belongs to the kernel of $d\bar{\partial} F$, hence is the real part of holomorphic function; this is $U(2)$-invariant, hence constant. \qed

Definition 16. A Kähler metric $(g, J, \omega)$ defined on a Hirzebruch-like ruled surface $M = \mathbb{P}(1 \oplus L)$ is called admissible if its restriction to $M_0$ admits a globally defined Kähler potential, $F = F(t)$, which only depends on $t$; on $M_0$, we then have

$$\omega = \omega_\psi := \psi \, dd^c t + \psi' \, dt \wedge d^c t,$$

by setting: $\psi(t) = F'(t)$, and $\psi'(t)$ denotes the derivative of $\psi$ with respect to $t$.

Notice that a 2-form $\omega_\psi$ of the form (10.3.5) is the Kähler form of a Kähler metric, $g_\psi$ say, defined on $M_0$ if and only if $\psi$ and $\psi'$ are both positive

By Theorem 3.5.1, Proposition 10.2.3 and Proposition 10.3.1, any extremal Kähler metric on a Hirzebruch surface $F_t$ is isomorphic, via an element of $\text{H}(F_t)$, to an admissible metric $g_\psi$.

If $M = \mathbb{P}(1 \oplus L)$ is a Hirzebruch-like ruled surface of genus greater than 0, then the maximal subgroups of $\text{H}(M)$ are conjugate to $S_1$ — cf. Proposition 10.2.3 — and we can no longer pretend a priori that all extremal Kähler metrics on $M$ can be made admissible by the action of $\text{H}(M)$. In this case, we agree to search a priori for extremal Kähler metrics among admissible Kähler metrics. Notice that the latter are obviously $S^1$-invariant.
For any admissible Kähler metric on $M_0$, the (positive, increasing) function $\psi$ has then the following geometric interpretation:

**Lemma 10.3.1.** Let $g_\psi$ be any admissible Kähler metric on $M_0$, whose Kähler form $\omega_\psi$ is given by (10.3.5) for some positive, increasing function $\psi$ defined on $(-\infty, +\infty)$. Then, $\omega_\psi$ is $S^1$-invariant for the natural $S^1$-action on $M_0 \subset M$ and $\psi$ is a momentum of this action with respect to $\omega_\psi$, meaning that:

\[(10.3.6) \quad \iota_T \omega_\psi = -d\psi,\]

where $T$ denotes the generator of the $S^1$-action.

Moreover, for any (positive) value $c$ of the momentum $\psi$ the corresponding Kähler quotient $\psi^{-1}(c)/S^1$ is $\Sigma$, equipped with the Kähler structure $c \omega_L$, of constant sectional curvature equal to $\frac{\kappa}{c}$.

**Proof.** The $S^1$-action on $M$ is induced by the standard $S^1$-action on $L$, which preserves the complex structure of $L$, hence of $M$, and any hermitian (fiberwise) inner product, in particular $h$, the radial function and $t = \log r$ on $M_0$, corresponding Chern connection and its curvature form $\pi^* \omega_L$: $\omega_\psi$ is thus preserved by this action. Moreover, from (10.3.3), (10.3.4) and $dt(T) = 0$, we infer: $\iota_T dt = 0$ and $\iota_T (dt \wedge dt) = -dt$; (10.3.6) follows readily.

For any $c > 0$, the level set $\psi^{-1}(c)$ is a distance sphere in $M_0$ and the restriction of $\omega_\psi$ to this sphere is equal to $c dd^c t = c \pi^* \omega_L = \pi^* (c \omega_L)$. □

Admissible Kähler metrics which are defined on the whole of $M$ are then characterized as follows:

**Proposition 10.3.2.** Let $g_\psi$ be any admissible Kähler metric on $M_0$, whose Kähler form $\omega_\psi$ is given by (10.3.5) for a positive, increasing function $\psi$. Then, $g_\psi$ extends to a Kähler metric to $\Sigma_0 \cup M_0$ if and only if the following condition is satisfied:

(A0) Near $t = -\infty$, $\psi(t) = \Phi_0(e^{2t})$, where $\Phi_0$ is a smooth positive function defined in some neighbourhood of $0$ in $\mathbb{R}^\geq 0$, with $\Phi_0(0) = a > 0$ and $\Phi_0'(0) > 0$.

Similarly, $g_\psi$ extends to $M_0 \cup \Sigma_\infty$ if and only if:

(A\infty) Near $t = +\infty$, $\psi(t) = \Phi_\infty(e^{-2t})$, where $\Phi_\infty$ is a smooth positive function defined in some neighbourhood of $0$ in $\mathbb{R}^\geq 0$, with $\Phi_\infty(0) = b > a > 0$ and $\Phi_\infty'(0) < 0$.

Then, $g_\psi$ extends to a Kähler metric on $M$ if and only conditions (A0) and (A\infty) are both satisfied. If so, we have in particular:

\[(10.3.7) \quad \lim_{t \to -\infty} \psi(t) = a, \quad \lim_{t \to +\infty} \psi(t) = b,\]

with $0 < a < b$, whereas $\psi'(t)$ and all successive derivatives tend to $0$ when $t$ tends to $\pm \infty$ in such a way that

\[(10.3.8) \quad \lim_{t \to -\infty} \frac{\psi'(t)}{\psi(t) - a} = 2, \quad \lim_{t \to +\infty} \frac{\psi'(t)}{\psi(t) - b} = -2,\]

and

\[(10.3.9) \quad \lim_{t \to -\infty} \frac{\psi^{(j+1)}(t)}{\psi^{(j)}(t)} = 2, \quad \lim_{t \to +\infty} \frac{\psi^{(j+1)}(t)}{\psi^{(j)}(t)} = -2,\]
where, for each integer \( j \geq 1 \), \( \psi^{(j)} \) denotes the \( j \)-th derivative of \( \psi \). Moreover, the Kähler class \([\omega_\psi]\) is then \( \Omega_{a,b} \) given by (10.2.5).

**Proof.** The Kähler metric smoothly extends to \( \Sigma_0 \) if and only if it is smooth defined on the total space of \( L = M \setminus \Sigma_\infty \). By setting \( \psi(t) = \Phi_0(e^{2t}) \), \( \omega_\psi \) can be written as \( \omega_\psi = \Phi_0(r^2)\pi^*\omega_2 + 2\Phi'_0(r^2)dr \wedge d\sigma \), where both \( \pi^*\omega_2 = dd^*t \) and \( dr \wedge d\sigma \) are smoothly defined on \( L \). We conclude that \( \omega_\psi \) is smoothly defined on \( \Lambda \) if and only if \( \Phi_0 \) and \( \Phi'_0 \) are smooth and positive on \( \mathbb{R}^{\geq 0} \). In particular \( a := \lim_{t \to -\infty} \psi(t) = \Phi_0(0) \) is positive as well as \( \lim_{t \to -\infty} \psi'(t)e^{-t} = \Phi'_0(0) \). The second part of the proposition is obtained by exchanging the roles of \( \Sigma_0 \) and \( \Sigma_\infty \), by observing that \( M \) can also be identified to \( \mathbb{P}(1 \oplus L^*) \), i.e. as the natural compactification of the dual line bundle \( L^* \) over \( \Sigma \) obtained by adding a “infinity section”: in this picture, the latter is \( \Sigma_0 \), whereas \( \Sigma_\infty \) becomes the zero section of \( L^* \). (note that the two pictures are isomorphic via the transformation \( t \to -t \); this transformation changes the orientation and transforms \( J \) into the complex structure \( \tilde{J} \) obtained by reversing on the fibres of \( \pi \) and keeping \( J \) on the horizontal distribution determined by the Chern connection). Since \( \int_{\Sigma_0} \omega_L = 2\pi t \), we have that \( \int_{\Sigma_0} \omega_\psi = 2\pi t a \) and \( \int_{\Sigma_\infty} \omega_\psi = 2\pi t b \). In view of (10.2.3), this implies the last statement of Proposition 10.3.2. \( \square \)

Positive, increasing smooth functions \( \psi = \psi(t) \) defined on the real line \(( -\infty, +\infty )\) which satisfy the boundary conditions \((A_0)-(A_\infty)\) near \( t = \pm \infty \) for some pair of real numbers \( a, b \) will be referred to as admissible momentum functions relative to the pair \( a, b \) (then, \( a, b \) necessarily satisfy (10.2.6)).

Notice that if \( \psi \) is an admissible momentum function relative to \( a, b \) as above, for any real number \( c \), the function \( \psi_c(t) := \psi(t + c) \) is again an admissible momentum function relative to \( a, b \) as well and the corresponding admissible Kähler metric, \( g_{\psi_c} \), is then the image of \( g_\psi \) by the (holomorphic) automorphism of \( M \) induced by the multiplication by \( e^c \) in \( L \). To avoid this ambiguity, it may be convenient to normalize \( \psi \) by fixing \( \psi(0) \) in \((a,b)\), e.g. \( \psi(0) = \frac{a+b}{2} \) or, in view of Remark 10.4.2, \( \psi(0) = \sqrt{ab} \).

For any pair \( a, b \) of real numbers satisfying (10.2.6), we denote by \( \mathcal{M}_{\Omega_{a,b}}^{adm} \) the space of admissible Kähler metrics of Kähler class \( \Omega_{a,b} \). In view of Proposition 10.3.2, \( \mathcal{M}_{\Omega_{a,b}}^{adm} \) is naturally identified with the space of admissible momentum functions relative to \( a, b \). Moreover:

**Proposition 10.3.3.** For any pair \( a, b \) of real numbers satisfying (10.2.6), the space \( \mathcal{M}_{\Omega_{a,b}}^{adm} \) is non-empty.

**Proof.** The function \( \psi_0 \) defined by

\[
\psi_0(t) = \frac{a + be^{2t}}{1 + e^{2t}},
\]

evidently satisfies the required conditions. \( \square \)

**Remark 10.3.1.** The Kähler metric, \( g_{\psi_0} \), determined by the standard admissible momentum function \( \psi_0 \) will be called the standard admissible Kähler metric in the Kähler class \( \Omega_{a,b} \). A Kähler potential for the Kähler
class \( \omega_{\psi_0} \) is the function \( F_0 = F_0(t) \) defined by
\[
F_0(t) = at + \frac{(b-a)}{2} \log (1 + e^{2t})
\]
\[
= bt + \frac{(b-a)}{2} \log (1 + e^{-2t})
\]
\[
= \frac{b}{2} \log (1 + e^{2t}) - \frac{a}{2} \log (1 + e^{-2t}).
\]
(10.3.11)

In restriction to any fiber \( \pi^{-1}(y) \) of \( \pi \), for any point \( y \) in \( \Sigma \), we have that \( \ddc_{\pi^{-1}(y)} = 0 \): a Kähler potential for the restriction of \( g_{\psi_0} \) to \( \pi^{-1}(y) \) is then \( \frac{b-a}{2} \log (1 + e^{2t}) = \frac{(b-a)}{2} \log (1 + r^2) \). We recognize a Kähler potential for the Fubini-Study metric of sectional curvature \( c = \frac{2}{(b-a)} \) of the projective line \( \mathbb{P}(\mathbb{C} \oplus L_y) \) determined by the hermitian inner product \( h_y \) on the affine open set \( \mathbb{P}(\mathbb{C} \oplus L_y) \mid [L_y] \), cf. (6.2.5) in Section 6.2. In view of this observation and of Remark 10.2.1, the standard admissible Kähler metric can be constructed according to the following recipe:

1. Realize \( M \) as the associated bundle \( M = Q \times_{S^1} \mathbb{P}^1 \) — cf. footnote 3 on page 253 — relatively to the standard \( S^1 \)-action on the complex projective line \( \mathbb{P}^1 \), where, we recall, \( Q \) denotes the set of elements of \( L \) of norm 1, viewed as a \( S^1 \)-principal bundle over \( \Sigma \). Since \( Q \times_{S^1} \mathbb{P}^1 \) is tautologically identified with \( (L \setminus \Sigma_0) \times_{\mathbb{C}^*} \mathbb{P}^1 \), cf. Remark 10.2.1, the manifold \( M \) constructed that way comes equipped with a natural complex structure.

2. Equip the complex projective line \( \mathbb{P}^1 \) with the standard Fubini-Study Kähler structure \( (g_{FS,c}, J_0, \omega_{FS,c}) \) of sectional curvature \( c = \frac{2}{(b-a)} \). Since \( g_{FS,c} \) is \( S^1 \)-invariant, each fiber \( \pi^{-1}(y) = \mathbb{P}(\mathbb{C} \oplus L_y) \) is therefore endowed with a Kähler metric, namely the Fubini-Study metric of sectional curvature \( c = \frac{2}{(b-a)} \) determined by the hermitian inner product of \( \mathbb{C} \oplus L_y \) induced by the hermitian inner product, \( h_y \), of \( L_y \).

3. By using the horizontal distribution, say \( H^\nabla \), on \( M = Q \times_{S^1} \mathbb{P}^1 \) induced by the Chern connection \( \nabla \), cf. footnote 3 of page 253, for any \( y \) in \( \Sigma \) and any point \( x \) in the fibre \( \pi^{-1}(y) = \mathbb{P}(\mathbb{C} \oplus L_y) \), decompose the tangent space \( T_x M \) into the direct sum
\[
T_x M = T_x(\mathbb{P}(\mathbb{C} \oplus L_y)) \oplus H^\nabla_y,
\]
(10.3.12)
where \( H^\nabla_y \) is naturally identified with \( T_y \Sigma \), via the differential \( \pi_* \) of \( \pi \). The standard admissible Kähler metric \( g_{\psi_0} \) is then determined by the following conditions:

- the two summands in (10.3.12) are \( g_{\psi_0} \)-orthogonal,
- the restriction of \( g_{\psi_0} \) to the horizontal part \( H^\nabla_x = T_y \Sigma \) is equal to \( \psi_0(x) g_L(y) \), where, we recall, \( g_L \) denotes the metric of \( \Sigma \) of constant sectional curvature \( \kappa \).
- the restriction of \( g_{\psi_0} \) to each fiber coincides with the natural Fubini-Study metric \( g_{FS,c} \), with \( c = \frac{2}{(b-a)} \), cf. (2).

More generally, any admissible Kähler metric \( g_\psi \) on \( M \) can be constructed according to the same recipe by only replacing \( \psi_0 \) by \( \psi \), in particular, the
Fubini-Sudy metric $g_{FS,c}$ on $\mathbb{P}^1$ with the $S^1$-invariant Kähler metric

\begin{equation}
\label{metric-CP1}
g_{\psi}^{\mathbb{P}^1} := \frac{\psi'(t)}{\psi_0'(t)} g_{FS,c},
\end{equation}

in (2) and (3). In (10.3.13), $t = \log r$ is viewed as a function defined on $\mathbb{C}^* = \mathbb{P}^1 \setminus (x_0 \cup x_{\infty})$, where $x_0 = (0 : 1), x_{\infty} = (1 : 0)$ denote the two fixed points of the natural $S^1$-action: $\zeta \cdot (u_1 : u_2) = (\zeta u_1 : u_2)$, and $r$ denotes the standard distance to the origin in $\mathbb{C} = \mathbb{P}^1 \setminus x_{\infty}$, so that $r((u_1 : u_2)) = \frac{|u_1|}{|u_2|}$. Then, $\psi' = \psi'(t)$ and $\psi_0' = \psi_0'(t)$ can both be viewed as defined on $\mathbb{P}^1 \setminus (x_0 \cup x_{\infty})$; in view of the boundary conditions of Proposition 10.3.2, $\frac{\psi}{\psi_0} = \frac{\psi(t)}{\psi_0(t)}$ then extends to a positive, smooth function on $\mathbb{P}^1$. When $\psi$ runs over the space of admissible momentum functions relative to the pair $a,b$, then $\frac{\psi'}{\psi_0'}$ can be any positive, $S^1$-invariant function, on $\mathbb{P}^1$, with the only constraint that $\int_{\mathbb{P}^1} \frac{\psi'}{\psi_0'} \omega_{FS,c} = \int_{\mathbb{P}^1} \omega_{FS,c} = 2\pi(b-a)$.

Via the above construction, admissible Kähler metrics $g_{\psi}$ on $M$ are entirely encoded by the Kähler metric $g_{\psi}^{\mathbb{P}^1}$ on the standard projective line $\mathbb{P}^1$, equipped with its standard complex structure and its standard $S^1$-action. On $\mathbb{C} = \mathbb{P}^1 \setminus (x_0 \cup x_{\infty})$, the Kähler form of $g_{\psi}^{\mathbb{P}^1}$ is $\omega_{\psi}^{\mathbb{P}^1} = \psi'(t) dt \wedge d^c t$. In particular, $\psi = \psi(t)$, now viewed as a function on $\mathbb{P}^1$, is a momentum relative to $\omega_{\psi}^{\mathbb{P}^1}$ of the natural $S^1$-action, which maps $\mathbb{P}^1$ to the closed interval $[a, b]$.

By using (1.19.6), we easily deduce that the scalar curvature of $g_{\psi}^{\mathbb{P}^1}$ has the following expression:

\begin{equation}
\label{scal-CP1}
s_{\psi}^{\mathbb{P}^1} = - \frac{(\log \psi'(t))''}{\psi'(t)}.
\end{equation}

Let $\psi$ be any admissible momentum function relative to the pair $a, b$. Since $\psi$ is an increasing function of $t$, $t$ can in turn be expressed as an increasing function of $\psi$, and we then set

\begin{equation}
\label{10.3.15}
\psi'(t) = \Theta(\psi),
\end{equation}

for some positive smooth function $\Theta = \Theta(x)$ defined for any $x$ in the open interval $(a, b)$. Following [109], $\Theta$ will be called the momentum profile of the metric $g_{\psi}$ and has the following geometric interpretation: the induced riemannian metric, say $g_{\psi, \Delta}$, on $(a, b)$ via the momentum map $\psi : M_0 \rightarrow (a, b)$ is given by

\begin{equation}
\label{10.3.16}
g_{\psi, \Delta} = dx \otimes dx \Theta(x).
\end{equation}

(here and henceforth, the natural parameter of the open interval $(a, b)$ is denoted by $x$). Alternatively, $\Theta(x)$ is equal to the square norm of $T$ at any point of $\psi^{-1}(x)$ in $M_0$.

In particular, $\Theta$ can also be regarded as the momentum profile of the Kähler structure $(g_{\psi}^{\mathbb{P}^1}, \omega_{\psi}^{\mathbb{P}^1})$ on $\mathbb{P}^1$ via the momentum map $\psi : \mathbb{P}^1 \rightarrow [a, b]$, cf. Remark 10.3.1. In terms of the momentum profile $\Theta$, the scalar curvature
Conversely, let \( \Theta = \Theta(\psi) \) be a smooth function defined on \((a,b)\), tending to \( +\infty \) as \( x \to +\infty \) and to \( -\infty \) as \( x \to -\infty \), when viewed as a function defined on \([a,b]\) via \( \psi \) has then the following expression (compare with (10.4.9) below):

\[
s_{\psi}^{p_{1}}(x) = -\Theta''(x).
\]

Momentum profiles associated to admissible momentum functions will be referred to as \textit{admissible momentum profiles} relative to the pair \( a, b \).

The momentum profile, \( \Theta_0 \), of the standard admissible metric \( \omega_{\psi_0} \) in Remark 10.3.1, called the \textit{standard admissible momentum profile} relative to \( a, b \), is easily checked to be given by

\[
\Theta_0(x) = \frac{2}{b-a}(x-a)(b-x).
\]

By using (10.3.17), we retrieve the fact that the scalar curvature of the corresponding Kähler metric on \( \mathbb{P}^1 \) is constant, equal to \( \frac{4}{b-a} \).

Notice that the momentum profile \( \Theta \) remains unchanged when \( \psi \) is replaced with \( \psi_c \) for any real number \( c \).

Admissible momentum profiles \( \Theta \) are characterized by the following proposition:

**Proposition 10.3.4.** By replacing the momentum function \( \psi \) with the momentum profile \( \Theta \), the space \( \mathcal{M}_{\Omega}^{\text{adm}} \) is identified, up to the action of \( \mathbb{R}^{\geq 0} \) on \( M \), with the space of smooth functions \( \Theta = \Theta(x) \) defined on the closed interval \([a,b]\) and satisfying the following properties

\[
\begin{align*}
(B_0) & \quad \Theta \text{ is positive on the open interval } (a,b); \\
(B_1) & \quad \Theta(a) = \Theta(b) = 0; \\
(B_2) & \quad \Theta'(a) = 2, \quad \Theta'(b) = -2.
\end{align*}
\]

**Proof.** (B0) follows from \( \psi' \) being positive on \(( -\infty, +\infty ) \); (B1) follows from the fact that \( \psi'(t) \to 0 \), when \( t \to \pm \infty \); (B2) follows from (10.3.8). Conversely, let \( \Theta = \Theta(x) \) be a smooth function defined on \([a,b]\), satisfying (B0)-(B1)-(B2) and let \( t = t(x) \) be defined on \((a,b)\) by

\[
ds \frac{dt}{dx} = \frac{1}{\Theta(x)};
\]

(10.3.19)

\( t(x) \) is then well-defined — up to translation by a constant — and increasing on \((a,b)\). Near \( x = a \), \( \Theta(x) = 2(x-a)\Theta_0(x) \), where \( \Theta_0 \) is positive, with \( \Theta_0(a) = 1 \); \( t \) is then of the form \( t(x) = \frac{1}{2} \log (x-a) + f_0(x) \), where \( f_0 \) is smooth at \( x = a \). Likewise, near \( x = b \), \( \Theta(x) = (b-x)\Theta_{\infty}(x) \), where \( \Theta_{\infty} \) is positive, with \( \Theta_{\infty}(b) = 1 \) and \( t \) is then of the form \( t(x) = -\frac{1}{2} \log (b-x) + f_\infty(x) \), where \( f_\infty \) is smooth at \( x = b \). This determines \( t \) as an increasing real function of \( x \), smoothly defined on the open interval \((a,b)\), tending to \(-\infty \) like \( \frac{1}{2} \log (x-a) \) when \( x \) tends to \( a \) and towards \( +\infty \) like \( -\frac{1}{2} \log (b-x) \) when \( x \) tends to \( b \). This in turn defines \( x \) as a well defined increasing function of \( t \) on the interval \(( -\infty, +\infty ) \), tending towards \( a \) when \( t \) tends to \( -\infty \) and to \( b \) when \( t \) tends to \( +\infty \), and smooth as a function of \( e^{\pm 2t} \) in the neighbourhood of \( t = \mp \infty \), hence satisfying conditions \( (A_0) - (A_{\infty}) \) of Proposition 10.3.2. \( \square \)
Remark 10.3.2. By choosing the momentum \( \psi \) as an independent variable, instead of \( t \) — this can be done, as \( \psi \) is an increasing function of \( t \) — \( \omega_\psi \) can be rewritten as

\[
\omega_\psi = \psi \pi^* \omega_L + d\psi \wedge \eta,
\]

where we have written \( \eta \) instead of \( d^c t \) to emphasize that \( d^c t \) is here viewed as the connection 1-form of the Chern connection 1-form of \( (L,h) \), with no explicit reference to \( t \) any longer. The corresponding metric, \( g_\psi \), can then be written as

\[
g_\psi = \psi \pi^* g_L + \frac{1}{\Theta(\psi)} d\psi \otimes d\psi + \Theta(\psi) \eta \otimes \eta,
\]

where \( \Theta = \Theta(\psi) \) is the momentum profile, defined on the interval \((a,b)\).

This change of viewpoint is best understood by introducing the symplectic potential, \( G = G(\psi) \), determined, up to an additive constant, by

\[
dG \frac{d\psi}{d}\psi = t.
\]

We then have: \( \frac{d^2 G}{d\psi^2} = \frac{1}{\Theta} \), so that the induced metric on the image of the momentum map \( \psi \) can be rewritten as \( \frac{d^2 G}{d\psi^2} d\psi \otimes d\psi \). Then, \( \psi = \frac{dF}{dt} \) and \( t = \frac{dG}{d\psi} \) appear as dual variables and the Kähler potential \( F = F(t) \) and the symplectic potential \( G = G(\psi) \) are linked together by the following partial Legendre transformation:

\[
F + G - t\psi = 0.
\]

Remark 10.3.3. In terms of (the Poincaré dual of) \( \Sigma_0 \) and \( \Sigma_\infty \), the first Chern class \( c_1(M) \) of a Hirzebruch-like ruled surface, whose general expression for a complex ruled surface is given by (10.1.4), has the following expression

\[
c_1(M) = (1 - \kappa) \Sigma_0 + (1 + \kappa) \Sigma_\infty.
\]

An alternative derivation of (10.3.24) uses the fact that \( c_1(M) \) is represented by \( \frac{1}{\pi^2} [\rho] \) — cf. Section 1.19 — where \( \rho \) stands for the Ricci form of any Kähler metric, in particular any admissible Kähler metric on \( M \). Let \( g_\psi \) be any such metric and recall that, on the open set \( M_0 \), its Ricci form has the form \( \rho = d^c w \), where \( w = w(t) \) is given by (10.4.4) in Lemma 10.4.2. We then have \( c_1(M) = (-x \Sigma_0 + y \Sigma_\infty) \), where \( x \), resp. \( y \), is the limit of \( w'(t) = \kappa - \frac{1}{2} \psi' - \frac{1}{2} \psi'' \) when \( t \) tends to \( -\infty \), resp. \( +\infty \). From the boundary conditions \((A_0)- (A_\infty)\), in particular from (10.3.8)-(10.3.9), we get that \( \lim_{t \to \pm \infty} \psi' = 0 \), whereas \( \lim_{t \to \pm \infty} \psi'' = \mp 2 \). We thus get (10.3.24) again.

By (10.3.24), \( c_1(M) \) belongs to the Kähler cone — so that \( M \) is Fano — if and only if \( \int_{\Sigma_0} c_1(M) \), \( \int_{\Sigma_\infty} c_1(M) \) and \( \int_{\Sigma} c_1(M) \) are all positive; by (10.3.24) this happens if and only if \( \kappa > 1 \), hence \( \kappa = 2 \) and this occurs if and only if \( g_0 = 0 \) and \( \ell = 1 \), i.e. \( M = \mathbb{F}_1 \). This, of course, is coherent with the classification of Fano surfaces given in Section 6.6.
10.4. Extremal admissible Kähler metrics

From now on we assume that $\omega_\psi$ is the Kähler form of an admissible Kähler metric $g_\psi$ defined on a Hirzebruch-like ruled surface $M = \mathbb{P}(1 \oplus L)$ of any genus $g$ and any degree $\ell > 0$, for some admissible momentum $\psi$.

The geometry of these Kähler metrics is easily described. We start with the following easy lemma:

**Lemma 10.4.1.** For any function $f = f(t)$ on $M$ which factorizes by $t$, the square norm $|df|^2$ and the laplacian $\Delta f$ of $f$ with respect to $g_\psi$ are given by

\begin{align}
|df|^2 &= \frac{(f')^2}{\psi'}, \\
\Delta f &= -\Lambda dd_c f = -\left(\frac{f'}{\psi} + \frac{f''}{\psi'}\right),
\end{align}

where $f'$, $f''$ denote the first and second derivative of $f$ with respect to $t$.

**Proof.** Since $f$ is a function of $t$ only, we have that $df = f' dt$, whereas we readily infer from (10.3.5) that $|dt|^2 = \frac{1}{\psi}$: this gives (10.4.1). Similarly, we have that $dd_c f = f' d\bar{d} t + f'' dt \wedge d\bar{t}$; since $\Lambda dd_c f = \frac{2d\bar{d} f \omega_\psi}{\omega_\psi \wedge \omega_\psi}$, we readily get (10.4.2). \(\square\)

**Lemma 10.4.2.** On $M_0$, the Ricci form, $\rho = \rho_\psi$, and the scalar curvature, $s = s_\psi$, of $\omega_\psi$ have the following expressions:

\begin{align}
\rho &= dd_c w = w'd\bar{d} t + w'' dt \wedge d\bar{t}, \\
\rho &= \frac{\psi}{\psi'} \left(\kappa - \frac{1}{2} \log \psi(t) - \frac{1}{2} \log \psi'(t)\right), \\
\rho &= 2(2w')' \psi', \\
\rho &= -\frac{1}{2} dd_c \log \frac{v_g}{v_0},
\end{align}

where $v_g$ is the volume form of $g$ and $v_0$ the volume form of the standard flat Kähler metric determined by any (local) holomorphic coordinate system. From (10.3.5), we readily infer

\begin{align}
v_g &= 2\psi\psi' d\bar{d} t \wedge dt \wedge d\bar{t},
\end{align}

on $M_0$. In order to compute $v_0$, it is here convenient to choose holomorphic coordinates of the form $(z, \lambda)$, where $z$ stands for any (local) holomorphic coordinate defined on some open subset, $U$, of $\Sigma$, whereas $\lambda$ depends on the choice of a (local) nowhere vanishing holomorphic section, $s$, of $L$ defined on $U$ and is then defined by: $\xi = \lambda(\xi) s(\pi(\xi))$, for any $\xi$ in $M_0$, where $\pi$ denotes the natural (holomorphic) projection from $M$ to $\Sigma$; then,
$v_0 = (\frac{1}{2}dz \wedge d\bar{z}) (\frac{1}{2}d\lambda \wedge d\bar{\lambda})$. The first factor $\frac{1}{2}dz \wedge d\bar{z}$ is the pull-back of the Kähler form $\omega_0$ of a flat Kähler metric defined on $U$, which is then of the form $\omega_0 = e^{-2\phi} \omega_L$, for some positive function $\phi$ defined on $U$; by using (1.19.6) again, $\phi$ is determined by:

$$\rho \omega_L = \kappa \omega_L = \rho_0 - dd^c \log \phi = -dd^c \log \phi.$$  

It follows that $\frac{1}{2}dz \wedge d\bar{z} = |\Phi(z)|^2 e^{2\kappa t} dt \wedge d\bar{t}$, for some nowhere vanishing holomorphic function $\Phi$ of $z$, whereas $\frac{1}{2}d\lambda \wedge d\bar{\lambda} = |\lambda|^2 dt \wedge d\bar{t}$, up to additional terms involving the derivative of $|\xi|$, hence $dz$ and $d\bar{z}$, which disappear when multiplied by $\frac{1}{2}dz \wedge d\bar{z}$; we then eventually get:

$$v_0 = |\Phi(z)|^2 |\lambda|^2 e^{2\kappa t} dt \wedge d\bar{t},$$

where $\Phi$ is holomorphic and nowhere vanishing. By plugging (10.4.7) and (10.4.8) into (10.4.6), we get (10.4.3). We then infer (10.4.5) from (10.4.2).

It is convenient to express the scalar curvature $s$ as a function of $\psi$ by using the momentum profile $\Theta$; we then have (cf. [109, Theorem A]):

**Lemma 10.4.3.** As a function of $\psi$, the scalar curvature of $\omega_\psi$ has the following expression:

$$s = \frac{2\kappa - (\psi \Theta(\psi))''}{\psi},$$

where $(\psi \Theta(\psi))''$ denotes the second derivative iwith respect to $\psi$.

**Proof.** Easy consequence of (10.4.5) (notice that $\Theta'(\psi) = \frac{\psi''}{\psi}$; more generally, any derivative with respect to $\psi$ is equal to the derivative with respect to $t$ divided by $\psi' = \Theta(\psi)$).

**Proposition 10.4.1 (E. Calabi [45]).** For any Hirzebruch-like ruled surface, an admissible Kähler metric on $M_0$ is extremal if and only if the scalar curvature $s$ is of the form

$$s = \alpha \psi + \beta,$$

for real constants $\alpha$, $\beta$. This happens if and only if there exist real constants $\gamma$, $\delta$, such that

$$\Theta(\psi) = \frac{P(\psi)}{\psi},$$

or, equivalently,

$$\psi \psi' = P(\psi),$$

where $P$ is the polynomial defined by

$$P(x) = \frac{-\alpha}{12} x^4 + \frac{\beta}{6} x^3 + \kappa x^2 - \frac{\gamma}{6} x - \frac{\delta}{12}.$$  

**Proof.** Recall that a Kähler metric is extremal if and only if its scalar curvature $s$ is a Killing potential, i.e. the (unnormalized) momentum of a hamiltonian Killing vector field; by Lemma 10.3.1 $\psi$ is a Killing potential; since both $s$ and $\psi$ only depends on $t$, the corresponding symplectic gradient are proportional to each other; they are then both Killing if and only if they...
are multiple of each other by a constant; we thus get (10.4.10). By putting together (10.4.9) and (10.4.10), we get the following differential equation

\[(\psi \Theta(\psi))'' = 2\kappa - \alpha \psi^2 - \beta \psi,\]

which obviously integrates into \(\psi \Theta(\psi) = P(\psi),\) where \(P\) is the polynomial defined by (10.4.13) and \(\gamma, \delta\) are any two (real) constants.

Proposition 10.4.1 achieves the first step of Calabi’s construction, namely the description of all extremal admissible \(\text{Kähler} \) metrics on \(M_0\), depending upon 4 real parameters. The second and final step, namely the construction of (admissible) extremal Kähler metrics on the whole of \(M\), is summarized by the following theorem: 10.3.2. We

**Theorem 10.4.1.** Let \(M = \mathbb{P}(1 \oplus L)\) be any Hirzebruch-like ruled surfaces of genus \(g\) and degree \(\ell\). Then,

(i) If \(g = 0\) or \(g = 1\), any \(\text{Kähler} \) class \(\Omega_{a,b}\) contains an essentially unique admissible extremal \(\text{Kähler} \) metric.

(ii) If \(g > 1\), denote by \(x_\kappa\) the unique real root greater than one of the equation

\[(\kappa x^2 + 4x - \kappa)^2 - 4x ((2 - \kappa)x + (2 + \kappa))^2 = 0,\]

where \(\kappa\) is given by (10.3.1). Then, a \(\text{Kähler} \) class \(\Omega_{a,b}\) contains an admissible extremal \(\text{Kähler} \) metric if and only if

\[\frac{b}{a} < x_\kappa\]

and it is then essentially unique.

(iii) In all cases, the scalar curvature of these admissible extremal \(\text{Kähler} \) metrics is non-constant.

**Proof.** First notice that the expression “essentially unique” in (i) and (ii) means that \(\psi\) is well-defined up to translation of the parameter \(t\), i.e. that \(g_w\) is uniquely defined up to the action of \(\mathbb{R}^{\geq 0}\) on \(M\).

By Proposition 10.4.1, an admissible \(\text{Kähler} \) metric on \(M_0\) is extremal if and only if \(\Theta(\psi) = \frac{P(\psi)}{\psi}\), where \(P\) is the polynomial defined by (10.4.13). Moreover, this metric is the restriction of a — still extremal — \(\text{Kähler} \) metric on \(M\) if and only if the momentum profile \(\Theta\) satisfies Conditions \((B_0) - (B_1) - (B_2)\) of Proposition 10.3.4. Now, Condition \((B_1)\) implies \(P(a) = P(b) = 0\), whereas Condition \((B_2)\) then implies \(P'(a) = 2a\) and \(P'(b) = -2b\). We thus get the following constraints on the polynomial \(P\):

\[P(a) = 0, \quad P(b) = 0, \quad P'(a) = 2a, \quad P'(b) = -2b, \quad P''(0) = 2\kappa\]

(the additional fifth constraint simply means that the coefficient of \(x^2\) in \(P\) is \(\kappa\)). The polynomial \(P\) is actually entirely determined by Conditions (10.4.17), hence by the chosen Hirzebruch-like ruled surface \(M\), via the parameter \(\kappa\), and by \(a, b\), i.e. by the chosen \(\text{Kähler} \) class \(\Omega_{a,b}\). It will be denoted by \(P_{\Omega_{a,b}}\) and called the \textit{Calabi polynomial} or, following [6], the \textit{extremal polynomial}, of \(\Omega_{a,b}\). Notice that \(P_{\Omega_{a,b}}\) is defined for any \(\text{Kähler} \) class \(\Omega_{a,b}\) on any
Hirzebruch-like ruled surface, independently of the actual existence/non-existence of an extremal metric within $\Omega_{a,b}$. A simple computation shows that $P_{a,b}$ has the following explicit expression:

$$P_{a,b}(x) = \frac{(x-a)(b-x)((2(a+b)-\kappa(b-a))x^2+(4ab+\kappa(b^2-a^2))x+ab(2(a+b)-\kappa(b-a)))}{(b-a)(a^2+4ab+b^2)}.$$  

(10.4.18)

Equivalently, the (normalized) coefficients $\alpha, \beta, \gamma, \delta$ appearing in (10.4.13) are given by

$$\alpha = \frac{12(2(a+b)-\kappa(b-a))}{(b-a)(a^2+4ab+b^2)},$$

$$\beta = \frac{12(-(a^2+b^2)+\kappa(b^2-a^2))}{(b-a)(a^2+4ab+b^2)},$$

$$\gamma = ab\beta = \frac{12ab(-(a^2+b^2)+\kappa(b^2-a^2))}{(b-a)(a^2+4ab+b^2)},$$

$$\delta = a^2b^2\alpha = \frac{12a^2b^2(2(a+b)-\kappa(b-a))}{(b-a)(a^2+4ab+b^2)}.$$  

(10.4.19)

Conversely, $\Theta(\psi) := \frac{P_{a,b}}{\psi}$ clearly satisfies Conditions $(B_1)$-$(B_2)$ and satisfies Condition $(B_3)$ — hence determines an (extremal) admissible Kähler metric by Proposition 10.3.4 — if and only if the extremal polynomial $P_{a,b}(\psi)$ is positive or, equivalently, the degree 2 polynomial $p$ defined by $p(\psi) := ((2+\kappa)a+(2-\kappa)b)\psi^2+(\kappa(b^2-a^2)+4ab)\psi+((2+\kappa)a+(2-\kappa)b)ab$ remains positive for any $\psi$ in $(a,b)$.

If $g = 0$, then $0 < \kappa \leq 2$ and $p(\psi)$ is clearly positive for any $\psi$ in $(a,b)$; if $g = 1$, then $\kappa = 0$ and we get the same conclusion: this proves Part (i) of Theorem 10.4.1.

We now assume $g > 1$, so that $\kappa < 0$. By putting $x = \frac{b}{a}$, the discriminant of $p$ is equal to $a^4f_\kappa(x)$, with $f_\kappa(x) = ((\kappa x^2+4x-\kappa)^2-4x((2-\kappa)x+(2+\kappa)^2))^2$. Notice that (10.4.15) is the equation $f_\kappa(x) = 0$. It is easily checked that $f_\kappa(x)$ has a unique real zero, say $x_\kappa$, greater than 1: this readily follows from the fact that $f_\kappa$ and its derivatives $f'_\kappa, f''_\kappa, f'''_\kappa$ are all positive at infinity but negative at $x = 1$. Moreover, $x_\kappa$ is certainly greater than the unique zeros greater than 1 of $f'_\kappa, f''_\kappa, f'''_\kappa$, in particular than the one of $f'''_\kappa$, which is equal to $1 - \frac{6}{\kappa} + \frac{4}{\kappa^2}$; we then have

$$x_\kappa > 1 - \frac{6}{\kappa} + \frac{4}{\kappa^2}$$  

(10.4.20)

(recall that we are considering the case when $\kappa < 0$). If $1 < x < x_\kappa$, the discriminant of $p$ is negative and $P_{a,b}(\psi)$ is then positive whenever $\psi$ belongs to $(a,b)$: we then get an extremal Kähler metric within the Kähler class $\Omega_{a,b}$. If $x \geq x_\kappa$, $p$ has two real roots, whose product is equal to $ab$ and whose sum is equal to $\frac{-\kappa x^2-4x+\kappa}{(2-\kappa)x+(2+\kappa)}$: because of (10.4.20), the latter is positive; moreover, the two roots of $p$ are then positive; since their product is $ab$, one of them certainly belongs to $(a,b)$: the corresponding Kähler metric is then singular along the hypersurface determined by this value of $\psi$. This proves Part (ii) of Theorem 10.4.1.
By (10.4.19), $\alpha$ has the sign of $2(b + a) - \kappa(b - a)$ and is then is then positive in all cases, as $\kappa \leq 2$. The scalar curvature $s = \alpha \psi + \beta$ — cf. (10.4.10) — is then non-constant; this proves Part (iii) of Theorem 10.4.1.

Theorem 10.4.1 is due to E. Calabi for $g = 0$ and to C. Tønnesen-Friedman [194], [195] for $g > 1$; for $g = 1$, it can be viewed as a special case of theorems by A. Hwang [107] and by D. Guan in [95]. It can be rephrased as follows:

**Theorem 10.4.2.** For any Hirzebruch-like ruled surface $M = \mathbb{P}(1 \oplus L)$ of genus $g$ and of degree $\ell$, let $\Omega_{a,b}$ be a Kähler class on $M$. Then, $\Omega_{a,b}$ contains an admissible extremal Kähler metric if and only if its extremal polynomial $P_{\Omega_{a,b}}$ is positive on the open interval $(a,b)$. If $g = 0$ or $g = 1$, this condition is satisfied for all Kähler classes. If $g > 1$, this condition is satisfied if and only if $\frac{b}{g} < x_\kappa$, where $x_\kappa$ is the unique real solution greater than 1 of the equation (10.4.15).

**Remark 10.4.1.** Theorems 10.4.1-10.4.2 leaves open the question of the existence/non-existence of non admissible extremal Kähler metrics on Hirzebruch-like ruled surfaces of genus $g > 1$, cf. [195]. A full answer of this question will be given below in Section 10.9.

**Remark 10.4.2.** For any Hirzebruch-like ruled surface $M = \mathbb{P}(1 \oplus L)$, the chosen hermitian inner product $h$ on $L$ determines a bundle isomorphism from $M$ to $\mathbb{P}(L^* \oplus 1)$; if $\tau_h$ denotes the hermitian duality from $L$ to $L^*$, this map is defined by $x = (z,u) \mapsto (\tau_h(z), \bar{u})$, for any $z$ in $\mathbb{C}$, any $u$ in $L_y$ and any $y$ in $\Sigma$. By composing this map with the natural identification $\mathbb{P}(1 \oplus L) = \mathbb{P}(L^* \oplus 1)$, obtained by tensoring any $x$ in $\mathbb{P}(1 \oplus L_y)$ by $L_y^*$, we get an involution, $\iota$, in $M$, which is an anti-holomorphic diffeomorphism on each fiber $\pi^{-1}(y)$ of $\pi$. This involution exchanges $\Sigma_0$ and $\Sigma_\infty$, preserves $M_0$ and, on $M_0 = L \setminus \Sigma_0$, is simply the transformation $u \mapsto \frac{u}{r^2(u)}$. The image of the natural complex structure $J$ of $M$ by $\iota$ is the complex structure, say $\tilde{J}$, which coincides with $-J$ on each fiber of $\pi$ and with $J$ on the horizontal distribution determined by the Chern connection $\nabla$. In particular, $J$ and $\tilde{J}$ determine opposite orientations. If $g_\psi$ is an admissible Kähler metric on $M_0$, then the pair $(g_\psi, \tilde{J})$ determines a hermitian structure whose Kähler form, $\tilde{\omega}_\psi$, has the following expression on $M_0$:

\[
\tilde{\omega}_\psi = \psi(t) \, dt \wedge \phi'(t) \, dt \wedge d^c t,
\]

(10.4.21)

where $d^c$ is still relative to $J$. Then, $\tilde{\omega}_\psi$ is no longer closed: we have instead

\[
d\tilde{\omega}_\psi = 2\psi \phi' dt \wedge d^c t = -2 \theta \wedge \tilde{\omega}_\psi,
\]

(10.4.22)

where $\theta$ — the so-called Lee form of the hermitian structure $(g, \tilde{J})$ — is given by

\[
\theta = -d \log \psi.
\]

(10.4.23)

It follows that $\tilde{\omega}_\psi := \psi^{-2} \tilde{\omega}_\psi$ is closed and, together with $\tilde{J}$, determines a Kähler structure, whose riemannian metric is $\psi^{-2} g_\psi$. The image by $\iota$ of
the Kähler structure \((\psi^{-2} g_\psi, \bar{J}, \psi^{-2} \bar{\omega}_\psi)\) is the admissible Kähler structure \((g_\phi, J, \omega_\phi)\), where \(\phi = \phi(t)\) is defined by

\[
\phi(t) = \frac{1}{\psi(t)}.
\]

Moreover, if \(g_\psi\) is extremal, i.e., by Proposition 10.4.1, if \(\psi\) satisfies (10.4.12) for the polynomial \(P\) defined by (10.4.13), then \(\phi\) satisfies \(\phi\phi' = P^*(\phi)\) for the “dual polynomial” \(P^*\) defined by

\[
P^*(x) = -\frac{\delta}{12} x^4 - \frac{\gamma}{6} x^3 + \kappa x^2 - \frac{\beta}{6} x - \frac{\alpha}{12},
\]

hence is extremal as well. Recall --- cf. Section 5.5 --- that an oriented 4-dimensional riemannian manifold is called selfdual, resp. antiselfdual, if the selfdual Weyl tensor \(W^+\), resp. the antiselfdual Weyl tensor \(W^-\), vanishes identically and that a Kähler manifold of (real) dimension 4 is antiselfdual if and only if its scalar curvature vanishes identically. In particular, any antiselfdual Kähler surface is extremal. In the present case, when \(g_\psi\) is an extremal admissible metric associated to the polynomial \(P\), its scalar curvature is \(s_{g_\psi} = \alpha \psi + \beta\) and \(g_\psi\) is then antiselfdual if and only if \(\alpha = \beta = 0\). Similarly, \(s_{g_\phi} = \delta \phi + \gamma\), and \(g_\phi\) is antiselfdual if and only if \(\gamma = \delta = 0\). Now, \(t^* g_\phi = \psi^{-2} g_\psi\); since \(t\) reverse the orientation and \(W^\pm\) are conformally invariant, we infer that \(g_\psi\) is selfdual if and only if \(\gamma = \delta = 0\). If \(g_\psi\) extends to \(M\), then \(\psi\) satisfies Conditions \((A_0) - (A_\infty)\) of Proposition 10.3.2 for the pair \(a, b\) and it is easily checked that \(\phi\) then satisfies \((A_0) - (A_\infty)\) for the pair \(\frac{1}{\psi}, \frac{1}{\bar{\psi}}\). Then, \(g_\phi\) extends to \(M\) and \([\omega_\phi] = \Omega_{\frac{1}{\psi}, \frac{1}{\bar{\psi}}} = \frac{1}{\bar{\psi}} \Omega_{a,b}\) (this is coherent with the obvious observation that, in formulæ (10.4.19), the transformation \(a \mapsto \frac{1}{\bar{\psi}}, b \mapsto \frac{1}{\psi}\) induces the transformation \(\alpha \mapsto \delta, \beta \mapsto \gamma, \gamma \mapsto \beta, \delta \mapsto \alpha\)). From the uniqueness part of Theorem 10.4.1, we infer that \(\phi(t) = \frac{1}{\bar{\psi}} \psi(t + c)\), for some real number \(c\), i.e. that \(\psi(t + c) \psi(-t) = ab\). The momentum \(\psi = \psi(t)\) of any extremal admissible Kähler metric on \(M\), when normalized by \(\psi(0) = \sqrt{ab}\), then satisfies

\[
\psi(t) \psi(-t) = ab.
\]

### 10.5. Bach-flat Kähler metrics on Hirzebruch-like ruled surfaces: the Page metric

The Bach tensor of a riemannian manifold has been introduced in Chapter 5. A Kähler manifold whose Bach tensor vanishes identically will be called Bach-flat. According to Theorem 5.4.1, a Kähler manifold of (real) dimension 4 is Bach-flat if and only if it is extremal and the riemannian metric \(\tilde{g} := s^{-2}g\) is Einstein on the open set where the scalar curvature \(s\) is non-zero. This section is devoted to the search of Bach-flat Kähler metrics among extremal admissible Kähler metrics on Hirzebruch-like ruled surfaces.

**Lemma 10.5.1.** An extremal admissible Kähler metric on \(M_0\) determined by a polynomial \(P\) as in Proposition 10.4.1 is Bach-flat if and only if the (normalized) coefficients \(\alpha, \beta, \gamma, \delta\) of \(P\) satisfy the condition

\[
\alpha \delta - \beta \gamma = 0.
\]
Proof. All Kähler metrics described by Proposition 10.4.1 are extremal: in view of Theorem 5.5.1 such a metric is then Bach-flat if and only the additional condition (5.4.15) is satisfied, i.e. if and only if $dd^c s + s \rho$ is proportional to the Kähler form $\omega$. Since the scalar curvature $s$ is a function of $t$, we have that $dd^c s = s'(t) dd^c t + s''(t) dt \wedge dd^c t$, whereas the Ricci form $\rho$ and the Kähler form $\omega$ are given by (10.4.3) and (10.3.5) respectively. Condition (5.4.15) then reads

\begin{equation}
\tag{10.5.2}
s' + sw' = s'' + sw''.
\end{equation}

Recall that $s = \alpha \psi + \beta$, that $w(t) = 2 t - \frac{1}{2} \log \psi - \frac{1}{2} \log \psi'$ and that $\psi \psi' = P(\psi)$, as the metric is extremal, where $P$ is given by (10.4.13), cf. (10.4.12). Notice that from $\psi \psi'' = P(\psi)$, we get

\begin{equation}
\tag{10.5.3}
\psi'' = \frac{P'(\psi) \psi'}{\psi} - \left( s' \right)^2 = \left( \frac{P'(\psi)}{\psi} - P(\psi) \right) \psi',
\end{equation}

We then get $s' = \alpha \psi' = \alpha \frac{P'(\psi)}{\psi}$, $s'' = \alpha \frac{\psi''}{\psi'} = \alpha \left( \frac{P'(\psi)}{\psi} - \frac{P(\psi)}{\psi'} \right)$, $w' = \kappa - \frac{1}{2} \frac{P'(\psi)}{\psi} \psi'' = - \frac{1}{2} \frac{P'(\psi)}{\psi} + \frac{1}{2} \frac{P'(\psi)}{\psi^2}$. By substituting in (10.5.2), this condition now reads

\begin{equation}
\tag{10.5.4}
2 \alpha P(\psi) - (2 \alpha \psi + \beta) P'(\psi) + \frac{\psi}{2} (\alpha \psi + \beta) P''(\psi) + \kappa (\alpha \psi + \beta) = 0.
\end{equation}

It turns out that all coefficients of the lhs of (10.5.4) — a polynomial of degree at most 4 — equal zero, except for the constant term, which is equal to $-\frac{1}{6} (\alpha \delta - \beta \gamma)$. \vskip 0.1in

Lemma 10.5.2. For any extremal e Kähler metric $g$ on $M_0$, associated to the polynomial (10.4.13), the scalar curvature $\tilde{s}$, of the riemannian metric $\tilde{g} := s^{-2} g$ — on the open set where the scalar curvature, $s$, of $g$ is not zero — is given by

\begin{equation}
\tag{10.5.5}
\tilde{s} = \beta^3 + 12 \kappa \alpha \beta + \alpha^2 \gamma + \frac{\alpha (\alpha \delta - \beta \gamma)}{\psi}.
\end{equation}

In particular, $\tilde{s}$ is constant if and only if either $\alpha = 0$, meaning that $s$ itself is constant, or $\alpha \delta - \beta \gamma = 0$, meaning that $g$ is Bach-flat by Lemma 10.5.1.

Proof. In general, for any two conformally related riemannian metrics $g$ and $\tilde{g} = \phi^{-2} g$, on a $n$-dimensional manifold, the scalar curvatures, $s$, of $g$ and $\tilde{s}$ of $\tilde{g}$ are related to each other by

\begin{equation}
\tag{10.5.6}
\tilde{s} = \phi^2 s - 2 (n - 1) \phi \Delta \phi - n (n - 1) |d \phi|^2,
\end{equation}
cf. e.g. \cite[Chapter 2]{28}. In the current case, where $\phi = |s|$ and $n = 4$, we thus get

\begin{equation}
\tag{10.5.7}
\tilde{s} = s^3 - 6 s \Delta s - 12 |ds|^2,
\end{equation}

From (10.4.1) and (10.4.2), and from (10.4.10)-(10.4.12), we infer

\begin{equation}
\tag{10.5.8}
|ds|^2 = \alpha^2 \frac{P(\psi)}{\psi}, \quad \Delta s = - \alpha \frac{P'(\psi)}{\psi}.
\end{equation}

Then, (10.5.5) follows readily, as well as the last statement of Lemma 10.5.2. \hfill \Box
Remark 10.5.1. The last statement of Lemma 10.5.2 is actually a quite general fact for extremal Kähler surfaces, in view of the following proposition, cf. [67]:

**Proposition 10.5.1.** For any extremal Kähler complex surface \((M,g,J)\), denote by \(M_0\) the open set where the scalar curvature, \(s\), does not vanish, and by \(M_1 \subset M_0\) the open set where \(s\) is not constant, i.e. where \(ds\) does not vanish. On \(M_0\), denote by \(\tilde{s}\) the scalar curvature of the conformal metric \(\tilde{g} = s^{-2}g\), by \(\tilde{r}\) its Ricci tensor, by \(\tilde{r}_0\) the trace-free part of \(\tilde{r}\). Then, \(ds\) and \(d\tilde{s}\) are related by

\[
(10.5.9) \quad d\tilde{s} = -12s \tilde{r}_0(ds).
\]

At any point of \(M_1\), we then have \(\tilde{r}_0 = 0\) if and only if \(d\tilde{s} = 0\). In particular, on \(M_1\), \(g\) is Bach-flat if and only if \(s\) is constant.

**Proof.** By definition of an extremal Kähler metric, \(ds\) is the \(g\)-dual of a (real) holomorphic vector field. By Lemma 1.23.4, we then have that \(r(ds) = \frac{1}{2} \Delta ds = \frac{1}{2} d(\Delta s)\). Equivalently, \(r_0(ds) = \frac{1}{2} d(\Delta s) - \frac{1}{2}s ds\). By using (5.4.16) and (10.5.7), we then infer (on \(M_0\)):

\[
s \tilde{r}_0(ds) = \frac{1}{2} s \Delta s - \frac{1}{4} s^2 ds + 2 (Dds, ds) + \frac{1}{2} \delta s ds
\]

\[
= \frac{1}{2} d(s \Delta s) - \frac{1}{12} ds^3 + d(\langle ds \rangle^2)
\]

\[
= -\frac{1}{12} d\tilde{s}.
\]

We thus get (10.5.9). Now, \(\tilde{r}_0\) is trace-free and, as an operator, commutes with \(J\): since \(n = 4\), this implies that at any point of \(M\), \(\tilde{r}_0\) is zero or an isomorphism. The second assertion then readily follows from (10.5.9). Finally, since \(g\) is extremal, it is Bach-flat on \(M_0\) if and only if \(\tilde{r}_0 = 0\); in view of the above, this happens on \(M_1\) if and only if \(s\) is constant. \(\square\)

**Theorem 10.5.1.** Let \(M = \mathbb{P}(1 \oplus L)\) be a Hirzebruch-like ruled surface of genus \(g\) and degree \(\ell\). Then:

(i) If \(g = 0\) and \(\ell = 1\), i.e. if \(M\) is the first Hirzebruch surface \(\mathbb{F}_1\) — equivalently \(M\) is the blow-up of \(\mathbb{F}^2\) at a point — then \(M\) admits a Bach-flat Kähler metric, say \(g\), unique up to scaling and the action of \(H(M)\); if \(\Omega_{a,b}\) is the corresponding Kähler class, then \(x := \frac{b}{a}\) is the unique real root greater than 1 of the equation

\[
(10.5.10) \quad (x^2 - 3)^2 - 16x = 0.
\]

Moreover, the scalar curvature, \(s\), of \(g\) is everywhere positive and the riemannian metric \(\tilde{g} := s^{-2}g\) is then an Einstein metric, of positive scalar curvature, globally defined on \(\mathbb{F}_1\), namely the Page metric, cf. Remark 10.5.2.

(ii) If \(g = 0\) and \(\ell = 2\), i.e. if \(M\) is the second Hirzebruch surface \(\mathbb{F}_2\), the scalar curvature of all extremal admissible Kähler metrics on \(M\) is positive everywhere but \(M\) admits no Bach-flat Kähler metric.

(iii) If \(g = 0\) and \(\ell > 2\), i.e. if \(M\) is the Hirzebruch surface \(\mathbb{F}_\ell\), with \(\ell > 2\), then \(M\) admits a Bach-flat Kähler metric, say \(g\), unique up to scaling.
and the action of $H(M)$; if $\Omega_{a,b}$ is the corresponding Kähler class, then $x := \frac{b}{a}$ is the unique real root greater than 1 of the equation

\begin{equation}
(10.5.11) \quad ((2 - \ell)x^2 - (2 + \ell))^2 - 4x((\ell - 1)x + (\ell + 1))^2 = 0.
\end{equation}

(iv) If $g > 0$, then $M$ admits an admissible Bach-flat Kähler metric, say $g'$, unique up to scaling and the action of homotheties on $M$. If $\Omega_{a,b}$ is the corresponding Kähler class, then $x := \frac{b}{a}$ is the unique real root greater than 1 of the equation

\begin{equation}
(10.5.12) \quad ((\kappa - 1)x^2 - (\kappa + 1))^2 - x((2 - \kappa)x + (\kappa + 2))^2 = 0,
\end{equation}

where $\kappa$ is defined by 10.3.1.

(v) In Cases (iii) and (iv), the scalar curvature, $s$, of $g$ vanishes on the distance sphere, $S_{\sqrt{ab}}$, determined by

\begin{equation}
(10.5.13) \quad \psi = \sqrt{ab}
\end{equation}

and $M \setminus S_{\sqrt{ab}}$ is the union of two connected open sets, namely $U_-$, where $\psi < \sqrt{ab}$, and $U_+$, where $\psi > \sqrt{ab}$. Denote by $\tilde{g}_-$, resp. $\tilde{g}_+$, the restriction of $s^{-2}g$ to $U_-$, resp. $U_+$; then, $\tilde{g}_- \text{ and } \tilde{g}_+$ are both complete, Einstein riemannian metric of negative scalar curvature; in particular, both are asymptotically hyperbolic (up to a common rescaling).

**Proof.** From Lemma 10.5.1 and from formulae (10.4.19) we readily infer that an extremal admissible Kähler metric on $M$ whose Kähler class is $\Omega_{a,b}$ is Bach-flat if and only if $x := \frac{b}{a}$ is a root of the equation (10.5.12), which is also the equation

\begin{equation}
(10.5.14) \quad p_\kappa(x) := (1 - \kappa)^2 x^4 - (2 - \kappa)^2 x^3 - 6x^2 - (2 + \kappa)^2 x + (1 + \kappa)^2 = 0,
\end{equation}

hence a polynomial equation of degree 4, except when $\kappa = 1$, i.e. when $g = 0$ and $\ell = 2$.

If $\kappa = 1$, i.e. if $g = 0$ and $\ell = 2$, the equation reduces to $p_1 := 4 - x(x + 3)^2 = 0$, which has no real solution greater than 1: indeed, for $x \geq 1$, $p'_1(x) = -3(x + 1)(x + 3)$ is negative, so that $p_1(x)$ is decreasing, with $p_1(1) = -12 < 0$. Moreover, by (10.4.19) with $\kappa = 1$, the restriction of the scalar curvature $s = \alpha \psi + \beta$ on $\Sigma_0$ is equal to $\frac{-12a+b}{(b-a)(a^2+4ab+b^2)}$, hence positive; since $\alpha > 0$ — cf. the proof of Theorem 10.4.1 (iii) — we infer that $s$ is positive everywhere. This achieves the proof of (ii).

We can then assume that $\kappa \neq 1$, i.e. that $p_\kappa$ is a polynomial of degree 4 with positive leading coefficient, whose successive derivatives are given by $p'_\kappa(x) = 4(1 - \kappa)^2 x^3 - 3(2 - \kappa)^2 x^2 - 12x - (2 + \kappa)^2$, $p''_\kappa(x) = 12(1 - \kappa)^2 x^2 - 6(2 - \kappa)^2 x - 12$, $p'''_\kappa(x) = 24(1 - \kappa)^2 x - 6(2 - \kappa)^2$. In particular, all these derivatives tend to $+\infty$ when $x$ tends to $+\infty$, whereas $p_\kappa(1) = -12 < 0$, $p'_\kappa(1) = -24 < 0$, $p''_\kappa(1) = 6(\kappa^2 - 4)$, $p'''_\kappa(1) = 6\kappa(3\kappa - 4)$. It readily follows that $p_\kappa$ has a unique real root greater than 1, say $x_\kappa$.

If $\kappa = 2$, i.e. if $g = 0$ and $\ell = 1$, (10.5.12) specializes to (10.5.10), whose unique solution greater than 1 satisfies $x^2 > 3$, as $p_2(\sqrt{3}) < 0$; it then follows form (10.4.19) that all coefficients $\alpha, \beta, \gamma, \delta$ are positive, so that $s = \alpha \psi + \beta$ and $\tilde{s} = \beta^3 + 24\alpha\beta + \alpha^2\gamma$ are both positive. This achieves the proof of (i).
If $0 < \kappa < 1$, i.e. if $g = 0$ and $\ell > 2$, any extremal Kähler, a fortiori, any Bach-flat Kähler metric can be made admissible by a suitable action of $H(M)$, whereas (10.5.12) specializes to (10.5.11); this proves (iii).

If $\kappa \leq 0$, i.e. if $g > 0$, there is no guarantee a priori that a Bach-flat metric can be made admissible by the action of $H(M) = \mathbb{C}^\ast$. On the other hand, the above reasoning shows that an extremal admissible Kähler metric on $M$ admits an admissible is Bach-flat if and only if its Kähler class is $\Omega_{a,b}$, with which $x := \frac{\bar{b}}{a} = \hat{x}_\kappa$. In view of Theorem 10.4.1, it remains to prove that, if $\kappa < 0$, i.e. if $g > 1$, $\hat{x}_\kappa < x_\kappa$; for that, it is sufficient to check that $f_\kappa(\hat{x}_\kappa)$ is negative, where $f_\kappa$ is the function which appears in the proof of Theorem 10.4.1; since $\hat{x}_\kappa$ is a root of (10.5.12), we have that $f_\kappa(\hat{x}_\kappa) = h(x)$, by setting $h(x) := (\kappa x^2 - 4x - \kappa)^2 - 4((1 - \kappa)x^2 + (1 + \kappa))^2 = -(x - 1)((2 - \kappa)x - (2 + \kappa))((2 - 3\kappa)x^2 + 4x + 2 + 3\kappa)$; since $\kappa$ is now assumed to be negative, it is a simple matter to check that $h(x)$ is negative for any $x \geq 1$. This completes the proof of (iv).

In order to prove (v), we first observe, by using formulae (10.4.19), that the Bach-flat condition $\alpha \delta - \beta \gamma = 0$ on $M$ can be rewritten as

\[
(10.5.15) \quad \beta^2 = ab \alpha^2,
\]
as $\delta = a_2 b_2 \alpha$ and $\gamma = ab \beta$, and that $\beta$ is negative for any values of $a, b$ if $\kappa < 1$, whereas $\alpha$ is always positive, as we already observed; it follows that if $\kappa < 1$, the zero locus of the scalar curvature $s = \alpha \psi + \beta \gamma$ of any admissible Bach-flat metric $g_\psi$ is determined by $\psi = -\frac{\bar{b}}{a} = \sqrt{ab}$; denote it by $S_\sqrt{ab}$. Since $ds = \alpha d\psi$ does not vanish on $M_0$, in particular on $S_\sqrt{ab}$, $s$ can be chosen as a defining function of $S_\sqrt{ab}$, so that $\tilde{g} := s^{-2}g$ — which is Einstein as $g$ is Bach-flat — is complete on each of the two connected components, $\bar{U}_-$ and $\bar{U}_+$, of $M \setminus S_\sqrt{ab}$. In particular, the (constant) scalar curvature $\tilde{s}$ of $\tilde{g}$ is negative. This can also be checked directly by using (10.5.5), which, in the Bach-flat case, reduces to

\[
(10.5.16) \quad \tilde{s} = \beta^2 + 12 \alpha \beta + \alpha^2 \gamma.
\]

By using (10.4.19), $\gamma = ab \beta$ and (10.5.15), we thus get

\[
(10.5.17) \quad \tilde{s} = \beta \alpha (2ab \alpha + 12 \kappa) = \alpha \beta \frac{12((a + b)^2(4ab + \kappa(b^2 - a^2)))}{(b - a)(a^2 + 4ab + b^2)},
\]
which has the opposite sign of $\kappa \hat{x}_\kappa^2 + 4\hat{x}_\kappa - \kappa$ since $\alpha \beta$ is negative. To check that this expression is positive, it is sufficient to show that $\hat{x}_\kappa$ is smaller than the positive root, say $x_0$, of the equation $\kappa x^2 + 4x - \kappa = 0$. By eliminating $\kappa = \frac{4x_0}{1 + x_0^2}$ in $p_\kappa(x_0)$ we obtain $p_\kappa(x_0) = \frac{(x_0^2 + 4x_0 - 1)^2(x_0 - 1)^2}{(x_0 + 1)^2}$, which is positive; this, shows that $x_0 > \hat{x}_\kappa$ as required. Finally, $\tilde{g}_\pm$ and $\tilde{g}_\mp$ are both asymptotically of (negative) constant sectional curvature, i.e., up to a common rescaling, both asymptotically hyperbolic; this is because both are Einstein, whereas the norm, $|\tilde{W}|$, of their Weyl tensor relatively to $\tilde{g}_\pm$, is related to the norm, relative to $g$, of the Weyl tensor, $W$, of $g$, by $|\tilde{W}| = s^2 |W|$, which tends to 0 when $\psi$ tends to $\sqrt{ab}$ as $|W|$ is bounded. \hfill \Box

Remark 10.5.2. The Einstein metric $\tilde{g}$ has been firstly discovered and constructed — in a different setting — by D. Page in [158], and is thus usually called the Page metric. For a long time, it has remained the only...
known example of a hermitian, non-Kähler, Einstein metric defined on a compact (smooth) complex surface until the discovery of the Chen–LeBrun–Weber metric on the blow-up of $\mathbb{P}^2$ at two points, cf. Remark 5.5.2.

**Remark 10.5.3.** The condition $\alpha\delta - \beta\gamma = 0$, as a condition that the corresponding Calabi extremal Kähler metric be conformally Einstein appears in [49].

### 10.6. Weakly selfdual Kähler metrics and hamiltonian 2-forms

In general, a Kähler manifold $(M, g, J, \omega)$ of any (real) dimension $n = 2m > 2$ is said to be *Bochner-flat* if its Bochner tensor $W^K$ — see Appendix A — vanishes identically, and *weakly Bochner-flat* if $W^K$ is co-closed with respect to the Levi-Civita connection $D$ of $g$, meaning that

$$
\delta D W^K = 0,
$$

where $\delta$ denotes the codifferential with respect to $D$ applied to $W^K$ when the latter is regarded as a $\Lambda^2 M$-valued 2-form. It then follows from the general Bianchi identity $\delta D R = 0$ applied to the riemannian curvature $R$ — cf. 1.18.4 — and from (A.2.6) that (10.6.1) is equivalent to the following condition for the Ricci form $\rho$:

$$
D X \rho = \frac{1}{4(m+1)} (ds \wedge JX^\flat - d^c s \wedge X^\flat + 2 ds(X) \omega),
$$

for any vector field $X$, cf. [4] for details. At this point, it is convenient to introduce the *normalized Ricci form* $\tilde{\rho}$, defined by

$$
\tilde{\rho} = \rho - \frac{s}{2(m+1)} \omega,
$$

and to observe that the Ricci form $\rho$ satisfies (10.6.2) if and only if the normalized Ricci form $\tilde{\rho}$ is a solution to the following equation

$$
D X \varphi = \frac{1}{2} (d \text{tr} \varphi \wedge JX^\flat - d^c \text{tr} \varphi \wedge X^\flat),
$$

for any vector field $X$: here, the unknown $\varphi$ is a real $J$-invariant 2-form and, we recall, $\text{tr} \varphi$ stands for the trace of $\varphi$ with respect to $\omega$, i.e. $\text{tr} \varphi = (\varphi, \omega)$; equivalently, $\text{tr} \varphi$ is the trace of the associated hermitian operator $\Phi$, cf. Section 1.12. In general, a solution to (10.6.4) is called a *hamiltonian form*.

The above discussion can be summarized by the following statement:

**Proposition 10.6.1.** A Kähler manifold is weakly Bochner-flat if and only if its normalized Ricci form $\tilde{\rho}$ is a hamiltonian 2-form.

Hamiltonian 2-forms have nice properties — this is why we substituted $\tilde{\rho}$ to $\rho$ and Equation (10.6.4) to Equation (10.6.2) — in particular the following one. For any $J$-invariant 2-form $\varphi$, denote by $\sigma_r(\varphi)$, $r = 1, \ldots, m$, the $r$-th elementary symmetric function of the roots of the corresponding hermitian operator $\Phi$: in particular, $\sigma_1(\varphi) = \text{tr} \varphi$, whereas $\sigma_m(\varphi)$ is the pfaffian, $\text{pf} \varphi$, of $\varphi$, alternatively defined by $\text{pf} (\varphi) = \frac{\varphi^m}{\omega^m}$, cf. Section 1.12. We then have:

**Proposition 10.6.2 ([4] Proposition 3).** If $\varphi$ is a hamiltonian 2-form, the $\sigma_r(\varphi)$, $r = 1, \ldots, m$, are pairwise Poisson commuting Killing potentials.
For any Hamiltonian 2-form $\varphi$, the 2-form $\tilde{\varphi} := \varphi + \sigma_1(\varphi) \varphi$ is closed (easy consequence of (10.6.4)): it will be referred to as the associated closed 2-form of $\varphi$ and is characterized by the equation

$$D_X \tilde{\varphi} = \frac{1}{2(m+1)} (\text{dtr} \tilde{\varphi} \wedge JX^\flat - \text{d} \text{tr} \tilde{\varphi} \wedge X^\flat + 2 \text{d} \text{tr} \tilde{\varphi} \omega).$$

If the Kähler metric is weakly Bochner-flat the normalized Ricci form $\tilde{\rho}$ is a Hamiltonian 2-form and $\sigma_1(\tilde{\rho}) = \frac{s}{2(m+1)}$ is then a Killing potential. We then have:

**Proposition 10.6.3.** Weakly Bochner-flat Kähler metrics — a fortiori Bochner-flat Kähler metrics — are extremal.

When $M$ is a Kähler complex surface, the Bochner tensor coincides with the anti-selfdual Weyl tensor $W^-$, cf. Section A.4: Bochner-flat Kähler metrics are then selfdual Kähler metrics and weakly Bochner-flat Kähler metrics are then simply called weakly selfdual. By Proposition 10.6.3 weakly selfdual — a fortiori selfdual — Kähler complex surfaces are then extremal.

The order of a Hamiltonian 2-form $\varphi$ is defined via the following

**Proposition 10.6.4 ([3] Proposition 13).** Let $\varphi$ be a Hamiltonian 2-form defined on a (connected) Kähler manifold $(M, g, J, \omega)$, $\sigma_1(\varphi)$, $r = 1, \ldots, m$, the associated Killing potential as in Proposition 10.6.2, $K_r = \text{grad}_g \sigma_r(\varphi) = J \text{grad}_g \sigma_r(\varphi)$ the corresponding (Hamiltonian) Killing vector fields. Then, there exists an integer $\ell$, $0 \leq \ell \leq m$, such that $K_1, \ldots, K_\ell$ are independent on a dense open set $M_0$ of $M$.

Hamiltonian 2-forms of order 0 are simply parallel 2-forms; in particular, the Kähler form $\omega$ is a Hamiltonian 2-form of order 0. In the opposite case, when $\varphi$ is a Hamiltonian 2-form of maximal order $m$, the tangent bundle $TM$ is trivialized over $M_0$ by the $2m$ vector fields $K_1, \ldots, K_m, JK_1, \ldots, JK_m$ and $M$ is then a toric Kähler manifold of a very specific type, called orthotoric, whose local structure is described in [3, Section 3.4] (the main specific feature of orthotoric Kähler structure of complex dimension $m$ is that they locally depend on $m$ real functions of one real variable, while a general toric Kähler metric locally depends on one function of $m$ real variables). In the intermediate cases, when $0 < \ell < m$, $M_0$ is foliated by totally geodesic orthotoric complex submanifolds of complex dimension $\ell$, [4, Proposition 8 and Theorem 1].

Hamiltonian 2-forms appear in many contexts of the Kähler geometry, in particular in the general Calabi-like constructions described in [3], [5], [6], [7], hence also in the framework of the Hirzebruch-like ruled surfaces considered in this chapter.

A Hamiltonian 2-form, $\varphi$, defined on a Hirzebruch-like ruled $M$ equipped with an admissible Kähler metric $\omega_\psi$, is called admissible if the restriction corresponding closed 2-form $\tilde{\varphi} = \varphi + \sigma_1(\varphi) \omega_\psi$ on $M_0$ is of the form $\tilde{\varphi} = \text{d} \text{d}^c v$, where $v = v(t)$ is a real function which factors through $t$. We then have:

**Proposition 10.6.5.** Let $M = \mathbb{P}(1 \oplus L)$ be a Hirzebruch-like ruled surface of genus $g$ and degree $\ell$ and let $g_\psi$ be an admissible Kähler metric on $M$. Then, the space of admissible Hamiltonian 2-forms with respect to $g_\psi$
is generated by the Kähler form $\omega_\psi$ and by the Hamiltonian 2-form $\varphi_0$ of order 1 which, on $M_0$, is defined by

\[(10.6.6) \quad \varphi_0 = \psi \psi' dt \wedge d^c t = \psi d\psi \wedge d^c t.\]

**Proof.** Let $\varphi$ be any admissible hamiltonian 2-form with respect to $g_\psi$ and let $\tilde{\varphi} = \varphi + \sigma_1(\varphi) \omega_\psi$ be the associated closed 2-form. We then have that $\tilde{\varphi} = \psi' dd^c t + \psi'' dt \wedge d^c t$. The remaining part of the argument then requires some identities given by the following lemma, where $D$ denotes the Levi-Civita connection of the metric $g_\psi$.

**Lemma 10.6.1.** For any admissible Kähler metric $(g_\psi, \omega_\psi)$ on $M_0$, we have

\[(10.6.7) \quad D_X(dt \wedge d^c t) = \frac{1}{2\psi}(dt \wedge JX^\flat + X^\flat \wedge d^c t),\]

\[(10.6.8) \quad D_X(d\psi \wedge d^c t) = -\frac{\psi'}{2\psi^2}(dt \wedge JX^\flat + X^\flat \wedge d^c t) + dt(X)\left(\left(\frac{\psi'}{\psi}\right)^2 dt \wedge d^c t - \frac{\psi'}{\psi} d\psi \wedge d^c t\right),\]

for any vector field $X$ on $U$.

**Proof.** The dual 1-form, $T^\flat$, of the vector field on $M_0$ generated by the natural $S^1$-action is given by $T^\flat = \psi' d^c t$. Since $T^\flat$ is a Killing vector field, we have that $DT^\flat = \frac{1}{2}dT^\flat$. We then infer

\[(10.6.9) \quad Dd^c t = \frac{1}{2}dd^c t - \frac{1}{2} \frac{\psi''}{\psi'}(dt \otimes d^c t + d^c t \otimes dt),\]

i.e.

\[(10.6.10) \quad D_X d^c t = \frac{1}{2} L_X dd^c t - \frac{1}{2} \frac{\psi''}{\psi'}(dt(X) d^c t + d^c t(X) dt),\]

for any vector field $X$. Since $DJ = 0$, we also get

\[(10.6.11) \quad D_X dt = \frac{1}{2} L_J X dd^c t - \frac{1}{2} \frac{\psi''}{\psi'}(dt(X) dt + d^c t(X) d^c t).\]

From (10.6.10) and (10.6.11) we readily obtain (10.6.7). By using $D\omega_\psi = 0$ and (10.6.7) we now readily obtain (10.6.8). \hfill \Box

From (10.6.7) and (10.6.8), we infer the following expression for the covariant derivative of $\tilde{\varphi} = \psi' dd^c t + \psi'' dt \wedge d^c t$:

\[(10.6.12) \quad D_X \tilde{\varphi} = \left(\frac{\psi''}{2\psi} - \frac{\psi'\psi''}{2\psi^2}\right)(dt \wedge JX^\flat - d^c t \wedge X^\flat) + (\psi'' - \frac{\psi'\psi''}{\psi'}) dt(X) d^c t + (\psi''' - \psi'' + \psi''') dt(X) dt \wedge d^c t.\]
Now, $\varphi$ is hamiltonian if and only if $\tilde{\varphi}$ satisfies (10.6.5), with $m = 2$. In particular, we must have
\[
\frac{v''}{\psi} - \frac{v'\psi'}{\psi^2} = \frac{v''}{\psi} - v''\left(\frac{1}{\psi} + \frac{\psi''}{(\psi')^2}\right) + v'\frac{\psi''}{\psi^3},
\]
which is clearly solved by
\[
v' = A\psi^2 + B\psi,
\]
for some real constants $A, B$. By substituting this value of $v'$ in (10.6.12), we get
\[
D_X\tilde{\varphi} = \frac{\psi'}{2}(dt \wedge JX^\varphi - d\mu \wedge X^\varphi + 2d\mu(X)\omega_\psi).
\]
On the other hand, the trace of $\tilde{\varphi} = v'dd^c t + v'' dt \wedge d^c t$ with respect to $\omega_\psi = \psi dd^c t + \psi' dt \wedge d^c t$ is given by
\[
\text{tr} \tilde{\varphi} = \frac{v'}{\psi} + \frac{v''}{\psi},
\]
(cf. Lemma 10.4.1). In the current situation when $v'$ is given by (10.6.13) we then get
\[
\text{tr} \tilde{\varphi} = 3A\psi + 2B.
\]
It follows that $\varphi$ is a hamiltonian 2-form if and only if $\tilde{\varphi} = dd^c v$, with $v'(t) = A\psi^2(t) + B\psi(t)$, for any constants $A, B$. Equivalently, $\varphi = \tilde{\varphi} - \frac{v'\psi}{\psi^2} \omega_\psi = A\psi\psi' dt \wedge d^c t + \frac{P}{2} \omega_\psi$. This completes the proof of the first part of Proposition 10.6.5. The second part directly follows from (10.4.12). □

**Lemma 10.6.2.** For any extremal admissible Kähler metric $g_\psi$, the normalized Ricci form has the following expression
\[
\rho = \frac{\alpha}{6} P(\psi) dt \wedge d^c t + \frac{\beta}{12} \omega_\psi + \frac{\gamma}{12} \bar{\omega}_\psi,
\]
where, we recall, $\bar{\omega}_\psi := \frac{1}{\psi} dd^c t - \frac{\psi'}{\psi^2} dt \wedge d^c t$ is the (antiselfdual) Kähler form introduced in Remark 10.4.2.

In particular, $\rho$ is a hamiltonian 2-form — equivalently, $g_\psi$ is weakly selfdual — if and only if
\[
\gamma = 0.
\]

**Proof.** The Ricci form $\rho$ of any admissible Kähler metric $(g, \omega_\psi)$ on $M_0$ is of the form $\rho = w dd^c t + w'' dt \wedge d^c t$, with $w'(t) = \kappa - \frac{1}{2} \frac{\psi'}{\psi} - \frac{1}{2} \frac{\psi''}{\psi^3}$. If the Kähler form is extremal, we have that $\psi\psi' = P(\psi)$, where $P$ is the polynomial defined by (10.4.13) and we then have $w' = \kappa - \frac{1}{2} P'(\psi)$, $w'' = \frac{1}{2} \left(\frac{P'(\psi)}{\psi} - \frac{P''(\psi)}{\psi^2}\right)$. We then get
\[
\rho = \left(\frac{\alpha}{6} \psi + \frac{\beta}{4} + \frac{\gamma}{12\psi^2}\right) \psi dd^c t + \left(\frac{\alpha}{3} \psi + \frac{\beta}{4} - \frac{\gamma}{12\psi^2}\right) \psi' dt \wedge d^c t,
\]
hence
\[
\rho = \left(\frac{\beta}{12} + \frac{\gamma}{12\psi^2}\right) \psi dd^c t + \left(\frac{\alpha}{6} \psi + \frac{\beta}{12} - \frac{\gamma}{12\psi^2}\right) \psi' dt \wedge d^c t,
\]
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which is the same as (10.6.17). By Proposition 10.6.5, \( \psi \psi' dt \wedge dt \) and \( \omega_\psi \)
are both hamiltonian 2-forms, while \( \bar{\omega}_\psi \) is not: the rhs of (10.6.17) is then
a hamiltonian 2-form if and only if \( \gamma = 0 \). \( \square \)

**Remark 10.6.1.** It easily follows from (10.6.17) that the pfaffian of
the normalized Ricci form \( \tilde{\rho} \) of any Calabi extremal metric is given by

\[
\text{pf}(\tilde{\rho}) = \frac{1}{144} \left( \beta (2\alpha \psi + \beta) + \gamma \left( \frac{2\alpha}{\psi} - \frac{\gamma}{\psi^2} \right) \right).
\]

Then, \( \text{pf}(\tilde{\rho}) \) is a Killing potential if and only if it is an affine function of
\( \psi \), hence if and only if \( \gamma = 0 \) or, by Lemma 10.6.2, if and only if \( \tilde{\rho} \) is a
hamiltonian 2-form.

**Remark 10.6.2.** We here present a more direct argument for the last
statement of Lemma 10.6.2. For that, it is convenient to simultaneously
consider the (admissible, extremal) Kähler structure \( (g = g_0, J, \omega_\psi) \) and the
dual Kähler structure \( (\bar{g} = \psi^{-2}g, \bar{J}, \bar{\omega}_\psi) \) introduced in Remark 10.4.2. The
anti-selfdual Weyl tensor \( W^- \) of \( g \) relatively to the orientation determined
by \( J \) is then the selfdual Weyl tensor of \( \bar{g} \) — which is also the selfdual
Weyl tensor of \( g \) by conformal invariance — relatively to the orientation
determined by \( \bar{J} \). Now, the selfdual Weyl tensor of any Kähler complex
surface is given by (5.5.22) and we thus get the following expression for
\( W^- \):

\[
W^- = \frac{\kappa_g}{8} \bar{\omega}_\psi \otimes \bar{J} - \frac{\kappa_g}{12} \Pi^-
\]

where \( \bar{\omega}_\psi \) is the Kähler form (10.4.21) of the hermitian pair \( (g, \bar{J}) \) introduced
in Remark 10.4.2, and where we have set

\[
\kappa_g := \frac{s_{\bar{g}}}{\psi^2} = \frac{\delta}{\psi^3} + \frac{\gamma}{\psi^2}.
\]

Here, \( s_{\bar{g}} \) denotes the scalar curvature of \( \bar{g} = \psi^{-2}g \), which is given by

\[
s_{\bar{g}} = \frac{\delta}{\psi} + \gamma;
\]

this identity in turn can be obtained either from \( \bar{g} = \iota^*g_0 \) — cf. Remark
10.4.2 — so that \( s_{\bar{g}} = \iota^*s_{g_0} = \iota^*(\delta \phi + \gamma) = \delta + \frac{\delta}{N} \), or by using the general
formula (10.5.6), which here specializes to

\[
s_{\bar{g}} = \psi^2 s - 6\psi \Delta \psi - 12 |d\psi|^2,
\]

whereas, by (10.4.1) and (10.4.2),

\[
\Delta \psi = - \left( \frac{\psi'}{\psi} + \frac{\psi''}{\psi^2} \right) = - \frac{P'(\psi)}{\psi} \quad \text{and} \quad |d\psi|^2 = \psi' = \frac{P(\psi)}{\psi};
\]

we then get \( s_{\bar{g}} = \alpha \psi^3 + \beta \psi^2 + 6P'(\psi) - 12 \frac{P(\psi)}{\psi} \),
which clearly implies

(10.6.24) \( (\kappa_g \) is actually the so-called *conformal scalar curvature* of
the hermitian pair \( (g, \bar{J}) \), i.e. the scalar curvature of the canonical Weyl connection
attached to \( (g_0, \bar{J}) \), cf. e.g. [10]). In the rhs of (10.6.22), \( \Pi^- \) must be
interpreted as follows: for any pair \( X, Y \) of vector fields, \( \Pi^-(X, Y) \) is the
anti-selfdual part of \( X \wedge Y \), viewed as an endomorphism of the tangent space
via the metric \( g \); then, \( \Pi^- \) is clearly parallel with respect to the Levi-Civita
cconnection, \( D \), of \( g \). For convenience, we introduce auxiliary local sections,
\( \omega_2, \omega_3 \), of \( \Lambda^- M \) in such a way that \( \omega_1 := \bar{\omega}_\psi, \omega_2, \omega_3 \) form a (local, normalized)
direct orthonormal frame of \( \Lambda^- M \) and we denote by \( I_1 = J, I_2, I_3 \) the
corresponding almost-complex structures with respect to \( g \), so that \( I_1 I_2 = I_3, \)

\[
\text{pf}(\tilde{\rho}) = \frac{1}{144} \left( \beta (2\alpha \psi + \beta) + \gamma \left( \frac{2\alpha}{\psi} - \frac{\gamma}{\psi^2} \right) \right).
\]
$I_2I_3 = I_1$, $I_2I_3 = I_1$ and $\Pi^- = \frac{1}{2}(\omega_1 \otimes I_1 + \omega_2 \otimes I_2 + \omega_3 \otimes I_3)$. We then have the following general fact, cf. e.g. [10]:

**Lemma 10.6.3.** The covariant derivative of $\bar{\omega}_\psi$ with respect to the Levi-Civita connection, $D$, of $g$ is given by

$$D\bar{\omega}_\psi = -I_3\theta \otimes \omega_2 + I_2\theta \otimes \omega_3$$

where $\theta$ denotes the Lee form of the hermitian structure $(g, \bar{J}, \bar{\omega}_\psi)$.

**Proof.** It is equivalent to prove that the covariant derivative of $\bar{J} = I_1$ is given by $D\bar{J} = -I_3\theta \otimes I_2 + I_2\theta \otimes I_3$. Since the norm of $\bar{J}$ is constant, $D\bar{J}$ is necessarily of the form $\alpha \otimes I_2 + \beta \otimes I_3$, for some (smooth) real 1-forms $\alpha, \beta$. Moreover, since $\bar{J}$ is integrable, we have that $D_{\bar{J}X}\bar{J} = JD_XJ$, for any vector field $X$, cf. Proposition 1.1.2. We then have that $\alpha(X) = \beta(JX)$, i.e. $\beta = I_1\alpha$, so that $D\bar{J} = -I_3\theta \otimes I_2 + I_2\theta \otimes I_3$ for some real 1-form $\theta'$. By contracting this expression with respect to $g$, we infer $\delta\bar{\omega}_\psi = 2\theta'$. Since $\bar{\omega}_\psi$ is selfdual with respect to the orientation induced by $\bar{J}$, we infer that $d\bar{\omega}_\psi = -2\theta' \wedge \bar{\omega}_\psi$. This identifies $\theta'$ with the Lee form $\theta$ defined by (10.4.22).

From (10.6.25), we easily infer

$$DW^- = \frac{d\kappa_g}{12} \otimes \omega_1 \otimes I_1 - \frac{d\kappa_g}{24} \otimes \omega_2 \otimes I_2 - \frac{d\kappa_g}{24} \otimes \omega_3 \otimes I_3$$

(10.6.26)

$$- \frac{\kappa_g}{8} I_3\theta \otimes \omega_2 \otimes I_1 + \frac{\kappa_g}{8} I_2\theta \otimes \omega_3 \otimes I_1$$

$$- \frac{\kappa_g}{8} I_3\theta \otimes \omega_1 \otimes I_2 + \frac{\kappa_g}{8} I_2\theta \otimes \omega_1 \otimes I_3,$$

and, upon contracting the above,

$$\delta DW^- = -\frac{1}{12} I_1(d\kappa_g - 3\kappa_g \theta) \otimes I_1$$

(10.6.27)

$$+ \frac{1}{24} I_2(d\kappa_g - 3\kappa_g \theta) \otimes I_2 + \frac{1}{24} I_3(d\kappa_g - 3\kappa_g \theta) \otimes I_3.$$

It follows that the Kähler pair $(g, J)$ is weakly selfdual if and only if the 1-form $d\kappa_g - 3\kappa_g \theta$ is identically zero. From (10.4.23) and (10.6.23) we readily infer:

$$d\kappa_g - 3\kappa_g \theta = \gamma \frac{d\psi}{\psi^3},$$

(10.6.28)

which is evidently identically zero if and only if $\gamma = 0$.

When considering weakly selfdual Kähler metrics on Hirzebruch-like ruled surfaces, we get

**Theorem 10.6.1.** Let $M = \mathbb{P}(1 \oplus L)$ be a Hirzebruch-like ruled surfaceog genus $g$ and degree $\ell$.

(i) If $g = 0$ and $\ell = 1$, i.e. if $M$ is the first Hirzebruch surface $\mathbb{F}_1$, then $M$ admits a weakly selfdual Kähler metric, unique up to scaling and the action of $\text{H}(M)$, its Kähler class is $\Omega_{a,b}$ with

$$\frac{b}{a} = \sqrt{3},$$

(10.6.29)
and the momentum function of the admissible representative in $\Omega_{a,b}$ is given, up to translation $^4$ of $t$, by

$$\psi(t) = \sqrt{ab} \left( \frac{a + b e^{2t}}{b + a e^{2t}} \right)^{\frac{1}{2}}, \tag{10.6.30}$$

whereas the momentum profile is given by

$$\Theta(x) = \frac{(x^2 - a^2)(3a^2 - x^2)}{2a^2 x}, \tag{10.6.31}$$

for $x$ in the interval $[a, b = \sqrt{3}a]$. In particular, the associated Calabi polynomial is then

$$P(x) = \frac{1}{a^2} (x^2 - a^2)(3a^2 - x^2), \tag{10.6.32}$$

whose normalized coefficients are

$$\alpha = \frac{6}{a^2}, \quad \beta = \gamma = 0, \quad \delta = 18a^2. \tag{10.6.33}$$

The normalized Ricci form is then the Hamiltonian 2-form

$$\tilde{\rho} = \frac{1}{2a^2} (\psi^2 - a^2)(3a^2 - \psi^2) \, dt \wedge d^c t \tag{10.6.34}$$

and the Pfaffian $\text{pf}(\tilde{\rho})$ of $\tilde{\rho}$ is equal to 0.

(ii) If $g = 0$ and $\ell > 1$, i.e. if $M = F_\ell$ with $\ell > 1$, $M$ admits no weakly self-dual Kähler metric.

(iii) If $g > 0$, then $M$ admits no admissible weakly self-dual Kähler metric.

PROOF. By Lemma 10.6.2, an admissible Kähler metric $g_\psi$ is weakly self-dual if and only if $\gamma = 0$; from (10.4.19) this can only occur if $\kappa = 1$, hence $g = 0$ and $\ell = 1$, or $M = F_1$, and if $b^2 = 3a^2$. The remaining normalized coefficients of the Calabi polynomial $P(\psi)$ are then given by (10.6.33) and $P(\psi)$ itself by (10.6.32). Since, $P(\psi)$ is a quadratic function of $\psi^2$, the differential equation $\psi' = P(\psi)$ is easily solved by the function $\psi(t)$ given by (10.6.30), up to translation of $t$ and (10.6.31) is easily deduced from (10.6.30) or, more simply, by the fact that $\Theta(x) = \frac{P(x)}{x}$ for any extremal admissible Kähler metric. Finally, (10.6.34) is a direct consequence of (10.6.17), with $\beta = \gamma = 0$ and $\text{pf}(\tilde{\rho})$ readily follows from (10.6.34) or from (10.6.21). \qed

REMARK 10.6.3. An analytic argument for the existence of weakly self-dual Kähler metrics on $F_1$ appears in [111].

It turns out that statement (iii) in Theorem 10.6.1 can be replaced by: “If $g > 0$, then $M$ admits no weakly self-dual Kähler metric (admissible or not)” This follows either from Theorem 10.9.1, or from the following one (a weaker version of Theorem 10.6.2 previously appeared in [110]):

---

$^4$Here $t$ is chosen in such a way that $\psi(0) = \sqrt{ab}$, hence $\psi(t)\psi(-t) = ab$, cf. Remark 10.4.2.
Theorem 10.6.2 ([3] Theorem 5). A (smooth) compact weakly selfdual Kähler complex surface is either of constant scalar curvature — hence Kähler-Einstein or, locally, the product of two Riemann surfaces of constant Gauss curvature — or isomorphic to a Calabi weakly selfdual Kähler metric on the first Hirzebruch surface \( \mathbb{F}_1 \).

For the proof, we refer the reader to [3]. This result in turn can be compared with the following theorem by B.-Y. Chen [54], cf. also [67, Theorem 1]:

Theorem 10.6.3. A (smooth) compact selfdual Kähler complex surface is either of constant holomorphic sectional curvature or locally the product of two Kähler Riemann surfaces of opposite constant curvature and its generalization by R. Bryant [41] (first stated — with an incomplete argument however — in [113]; a revised version appears in [114]; an alternative argument can be found in [5]):

Theorem 10.6.4. A (smooth) compact Bochner flat Kähler manifold is either of constant holomorphic sectional curvature or locally the product of two Kähler manifolds of opposite constant holomorphic sectional curvature.

For the proof we refer the reader to the original papers.

Remark 10.6.4. In all results mentioned above it is essential to consider smooth Kähler manifolds, as there are many more compact examples if we allow for Kähler metrics with orbifold singularities. In particular it is shown in [41] — cf. also [63] — that each weighted projective space admits a Bochner flat Kähler metric, for any (relatively prime) weights \( a_0, a_1, \ldots, a_m \), whose scalar curvature is non-constant except in the standard case when the weights are 1, 1, \ldots, 1 and the metric is the Fubini-Study metric. Other examples appear in [5].

10.7. The Futaki character of admissible ruled surfaces

Define \( \tilde{T} \) by \( \tilde{T} := -JT \), where, we recall, \( T \) denotes the generator of the \( S^1 \)-action on the Hirzebruch-like ruled surface \( M = \mathbb{P}(1 \oplus L) \) induced by the natural \( S^1 \)-action on \( L \). Then \( \tilde{T} \) is a (real) holomorphic vector field, whose real potential with respect to any admissible metric \( g_\psi \) is equal to the momentum of \( T \), hence to \( \psi \), up to an additive constant. In this section, we compute \( \tilde{F}_{\Omega_{a,b}}(\tilde{T}) \), where \( \tilde{F}_{\Omega_{a,b}} = \frac{F_{\Omega_{a,b}}}{V_{\Omega_{a,b}}} \) denotes the (normalized) Futaki character of \( M \) with respect to the Kähler class \( \Omega_{a,b} \), cf. Section 4.12. We then have the following proposition, which is a particular case of [6, Proposition 6]:

Proposition 10.7.1. For any Hirzebruch-like ruled surface \( M \) of genus \( g \) and degree \( \ell \), and for any Kähler class \( \Omega_{a,b} \), we have

\[
(10.7.1) \quad \tilde{F}_{\Omega_{a,b}}(\tilde{T}) = \frac{2}{3} \frac{(b-a)}{(a+b)^2} \left( 2(a+b) - \kappa (b-a) \right),
\]

where \( \kappa \) is defined by (10.3.1). In particular,

\[
(10.7.2) \quad \tilde{F}_{\Omega_{a,b}}(\tilde{T}) > 0,
\]

for any genus \( g \) and any degree \( \ell \).
10.7. THE FUTAKI CHARACTER OF ADMISSIBLE RULED SURFACES

Proof. The real potential of $\tilde{T}$ with respect to an admissible metric $g_\psi$ is equal to the part of the momentum $\psi$ which integrates to 0, hence to $\psi - \bar{\psi}$. By the very definition of $\tilde{F}_{\Omega_{a,b}}$ — see Section 4.12 — we then have

$$\tilde{F}_{\Omega_{a,b}}(\tilde{T}) = \frac{1}{V_{a,b}} \int_M s_{g_\psi}(\psi - \bar{\psi}) v_{g_\psi} = \frac{1}{V_{a,b}} \int_M \bar{\psi} s_{g_\psi} v_{g_\psi} - \bar{s}_{g_\psi} \bar{\psi},$$

for any admissible Kähler metric $g_\psi$ in $\Omega_\psi$, where $V_{a,b}$ denotes the total volume of $M$ with respect to $\Omega_{a,b}$.

From (10.2.5), we readily infer that

$$V_{a,b} = \int_M \Omega_{a,b} \cdot \Omega_{a,b} = 2\pi^2 \ell (b^2 - a^2).$$

Similarly, by using (10.3.24), we infer that the total scalar curvature, $S_{a,b}$, determined by the Kähler class $\Omega_{a,b}$, is given by

$$S_{a,b} = 4\pi \int_M c_1(M) \wedge \Omega_{a,b}$$

$$= 8\pi^2 \ell ((a + b) + \kappa(b - a)).$$

The mean scalar curvature $\bar{s} := \frac{S_{a,b}}{V_{a,b}}$ is then given by

$$\bar{s} = \frac{4 ((a + b) + \kappa(b - a))}{(b^2 - a^2)}.$$

The computation of the rhs of (10.7.3) relies on the following simple general fact. Let’s call invariant a (real) function $f$ on $M$ which can be expressed on $M_0 = L \setminus (\Sigma_0 \cup \Sigma_\infty)$ as a function of $t$. We then have

**Lemma 10.7.1.** For any invariant real function $f = f(t)$ and for any admissible metric $g = g_\psi$, we have

$$\int_M f v_g = 4\pi^2 \ell \int_{-\infty}^{+\infty} f \psi \psi' dt.$$

**Proof.** This readily follows from

$$v_g = \psi \psi' dd^c t \wedge dt \wedge d^c t = \psi \psi' dt \wedge \eta \wedge \pi^* \omega_\Sigma,$$

for any admissible metric, and the fact that, by (10.3.3) the contribution of $dd^c t$ in the integral is equal to $2\pi$, whereas by (10.3.4), the contribution of $dd^c t$ is equal to $2\pi \ell$. \hfill $\Box$

From (10.4.5) and by using (10.7.7), we get

$$\int_M s_g \psi v_g = 4\pi^2 \ell \int_{-\infty}^{+\infty} 2\psi (w' \psi)' dt$$

$$= 4\pi^2 \ell 2w' \psi^2|_{-\infty}^{+\infty} - 4\pi^2 \ell \int_M 2w' \psi' dt.$$

We already observed in Remark 10.3.3 that $\lim_{t \to -\infty} w'(t) = \kappa - 1$ and $\lim_{t \to +\infty} w'(t) = \kappa + 1$. We then have $2w' \psi^2|_{-\infty}^{+\infty} = 2(b^2 - a^2) \kappa + 2(a^2 + b^2)$, whereas

$$\int_{-\infty}^{+\infty} 2w' \psi' dt = \int_{-\infty}^{+\infty} (2\kappa \psi \psi' - (\psi')^2 - \psi \psi'') dt$$

$$= \kappa \psi^2|_{-\infty}^{+\infty} - \psi \psi'|_{-\infty}^{+\infty} = \kappa(b^2 - a^2),$$

where $\kappa$ is the scalar curvature of the Kähler metric $g_\psi$. $\Box$
as \( \lim_{t \to \pm \infty} \psi' = 0 \); we thus get

\[
\int_M s_g \psi v_g = 4\pi^2 \ell ((b^2 - a^2) \kappa + 2(a^2 + b^2)).
\]

We similarly get

\[
\int \psi v_g = 4\pi^2 \ell \int \int_{-\infty}^{+\infty} \psi^2 \psi' \, dt = 4\pi^2 \ell \frac{(b^3 - a^3)}{3},
\]

so that, by using (10.7.4),

\[
\overline{\psi} = \frac{2}{3} \frac{(b^3 - a^3)}{b^2 - a^2}.
\]

By using (10.7.6), we then have

\[
s_{g_0} \overline{\psi} = \frac{8}{3} \frac{(b^3 - a^3)((a + b) + \kappa (b - a))}{(b^2 - a^2)^2}.
\]

By plugging (10.7.9), (10.7.12) and (10.7.4) in (10.7.3), we eventually get (10.7.1), where the rhs is clearly positive for any genus \( g \) and any degree \( \ell \) (recall that \( \ell \) is assumed to be positive).

\[\square\]

**Corollary 10.7.1.** An Hirzebruch-like ruled surface of any genus and any degree carries no Kähler metric of constant scalar curvature.

**Proof.** By Theorem 4.12.1, the existence of a Kähler metric of constant scalar curvature in \( \Omega_{a,b} \) implies that the corresponding Futaki character \( \mathcal{F}_{\Omega_{a,b}} \) is trivial. Because of (10.7.2), this cannot happen for admissible ruled surfaces.

\[\square\]

**Remark 10.7.1.** Formula (10.7.1) fits with the general formula given in [6] for all admissible projective bundles. Note that \( \mathcal{F}_{\Omega_{a,b}}(\bar{T}) \) can also be written as

\[
\mathcal{F}_{\Omega_{a,b}}(\bar{T}) = \frac{(b-a)^2(a^2 + 4ab + b^2)}{18(a+b)^2} \alpha,
\]

meaning that \( \mathcal{F}_{\Omega_{a,b}}(\bar{T}) \) is a positive multiple of \( \alpha \), which, we recall, is the (normalized) leading coefficient of the Calabi polynomial \( P_{\Omega_{a,b}} \), as given by (10.4.19). This too fits with [6].

**10.8. The extremal vector field of a Hirzebruch-like ruled surface**

For a Hirzebruch-like ruled surface \( M = \mathbb{P}(1 \oplus L) \) of genus \( g \) and degree \( \ell \), a natural maximal compact subgroup of the automorphism group \( H(M) = H_0(M) \) is \( G \), with \( G = U(2)/\mu_\ell \) if \( g = 0 \), and \( G = S^1 \) if \( g > 0 \), cf. Propositions 10.2.2 and 10.2.3. In all cases, the center of \( G \) is the \( S^1 \)-action induced by the natural \( S^1 \)-action on \( L \), whose generator is denoted by \( T \).

Notice that admissible Kähler metrics are \( G \)-invariant, so that \( \mathcal{M}_{adm}^G \) is a subspace of \( \mathcal{M}_{\Omega}^{11} \).

For any admissible metric \( g_\psi \) on \( M \), the center of the corresponding space of Killing potentials \( P_{g_\psi}^G \) for the Poisson bracket — cf. Section 4.13 — is then the space generated by \( \psi \) and the constant 1.

It follows that, for any Kähler class \( \Omega_{a,b} \) on \( M \), the Killing part of the scalar curvature is of the form \( A\psi + B \), where \( A, B \) are constants depending
on \( a, b \), and the extremal vector field \( Z_{\Omega_{a,b}}^{G} \) is then equal to \( AT \). More explicitly, we have

**Proposition 10.8.1.** For any Hirzebruch-like ruled surface \( M = \mathbb{P}(1 \oplus L) \) of genus \( g \) and degree \( \ell \), and for any Kähler class \( \Omega_{a,b} \), let \( G \) be a maximal compact subgroup of \( H_{\text{red}}(M,J) \) as above. Then, for any admissible Kähler metric \( g_\psi \) in \( \mathcal{M}_{\Omega_{a,b}}^{G} \) the Killing part of the scalar curvature of \( g_\psi \) is given by

\[
\Pi_g(s_{g_\psi}) = \alpha \psi + \beta = \frac{2\kappa - P''_{\Omega_{a,b}}(\psi)}{\psi},
\]

where \( \alpha, \beta \) are the first two normalized coefficients of the Calabi polynomial \( P_{\Omega_{a,b}} \), whose explicit expression is given by (10.4.18), and \( \kappa = \frac{2(1-g)}{\ell} \) is the coefficient of \( \psi^2 \) in \( R_{\Omega_{a,b}} \), cf. (10.4.17). Moreover, the extremal vector field is given by

\[
Z_{\Omega_{a,b}} = \alpha T.
\]

**Proof.** We already know that \( \Pi_g(s_{g_\psi}) = A \psi + B \), for some real constants \( A, B \). These are determined by the following two relations. The first one comes from the fact that the Futaki character can be computed by only considering the Killing part of \( s_{g_\psi} \); we then get

\[
\tilde{\mathcal{F}}_{\Omega_{a,b}}(\tilde{T}) = \frac{1}{V_{a,b}} \int_M s_{g_\psi}(\psi - \overline{\psi}) v_g
\]

\[
= \frac{1}{V_{a,b}} \int_M \Pi_g(s_{g_\psi})(\psi - \overline{\psi}) v_g
\]

\[
= A \int_M (\psi - \overline{\psi}) v_g
\]

where the last integral can be computed as we already did, by using (10.7.7) which readily implies

\[
\int_M \psi^r v_g = 4\pi^{2} \ell \frac{(b^{r+2} - a^{r+2})}{r + 2},
\]

for any real number \( r \neq -2 \), whereas

\[
\int_M \psi^{-2} v_g = 4\pi^{2} \ell \log \frac{b}{a}.
\]

We then easily get

\[
A = \frac{18(a + b)^2}{(b - a)^2(a^2 + 4ab + b^2)} \tilde{\mathcal{F}}_{\Omega_{a,b}}(\tilde{T}) = \alpha,
\]

cf. (10.7.13). The second relation comes from the fact that \( \int_M s_{g_\psi} v_g = \int_M \Pi_g(s_{g_\psi}) v_g \), as the reduced scalar curvature \( s_{g_\psi} \) integrates to 0. We then get

\[
\bar{\sigma} = A \overline{\psi} + B,
\]

where the mean values \( \bar{\sigma} \) and \( \overline{\psi} \) are given by (10.7.6) and (10.7.11) respectively, and \( A = \alpha \), given by (10.4.19). We then get \( B = \beta \). Finally, for any admissible Kähler metric \( g_\psi \) in \( \mathcal{M}_{\Omega_{a,b}}^{G} \) we have that \( Z_{\Omega_{a,b}}^{G} = \text{grad}_{g_\psi}(\Pi_g^{G}(s_{g_\psi}) = \alpha \text{grad}_{g_\psi}(\psi) = \alpha T \).
10.9. Extremal metrics on complex ruled surfaces

In this section we compute the restriction of the relative K-energy relative to $G$ — see Section 4.14 — to the space $\mathcal{M}_{\Omega_{a,b}}^{adm} \subset \mathcal{M}_{\Omega_{a,b}}^G$, of admissible Kähler metrics, for any chosen Kähler class $\Omega_{a,b}$ on any Hirzebruch-like ruled surface $M = \mathbb{P}(1 \oplus L)$.

For that, we shall use the identification of $\mathcal{M}_{\Omega_{a,b}}^{adm}$ with the space, call it $\mathcal{A}_{a,b}$, of admissible momentum profiles $\Theta$ relative to the pair $a, b$ described by Proposition 10.3.4.

We then have the following proposition, which is a particular case of [6, Proposition 7]:

**Proposition 10.9.1.** For any Hirzebruch-like ruled surface $M = \mathbb{P}(1 \oplus L)$ of genus $g$ and degree $\ell$ and for any Kähler class $\Omega_{a,b}$ on $M$, the restriction of the relative K-energy $E_{a,b}^G$ to $\mathcal{M}_{\Omega_{a,b}}^{adm}$ has the following expression (up to an additive constant):

\[
E_{a,b}^G(\Theta) = 4\pi^2 \ell \int_a^b \left( P_{\Omega_{a,b}}(x) \frac{\Theta(x)}{\Theta(x)} + x \log(\Theta(x)) \right) dx,
\]

where we recall, $P_{\Omega_{a,b}}$ denotes the extremal polynomial of the Kähler class $\Omega_{a,b}$ and and elements of $\mathcal{M}_{\Omega_{a,b}}^{adm}$ are represented by their momentum profile $\Theta$, viewed as a function defined on the open interval $(a, b)$.

**Proof.** Fix any $\omega_0$ in $\mathcal{M}_{\Omega_{a,b}}^{adm}$ — e.g. the standard admissible metric $\omega_0$ of $\Omega_{a,b}$, cf. Remark 10.3.1 — so that any $\omega_0$ in $\mathcal{M}_{\Omega_{a,b}}^{adm}$ can be written as $\omega = \omega_0 + dd^c \phi$, where $\phi = \phi(t)$ is normalized by $\int_M f_v g = 0$ for any variation $f := \dot{\phi} := \frac{d}{ds}|_{s=0} \phi_s$ of $\phi$, hence of $\omega_0$ in $\mathcal{M}_{\Omega_{a,b}}^{adm}$. On $M_0 = L \setminus (\Sigma_0 \cup \Sigma_\infty)$, $\omega_0$ is written as $\omega = \psi dd^c \ell + \psi' dt \wedge d^c t$, with $\psi = F_0 + \phi'$, where $F_0$ is a Kähler potential of $\omega_0$, still a function of $t$ (if $\omega_0$ is the standard admissible metric, then we can choose $F_0(t) = F_{\omega_0}$, where $F_{\omega_0}$ is given by (10.3.11)). The corresponding variation of $\psi$ is then $\dot{\psi} = f'$; similarly, $\dot{\psi'} = \dot{\psi} = f''$, etc. On the other hand, from the defining relation $\Theta(\psi) = \psi'$, we infer the following expression for the first variation of the momentum profile $\Theta$:

\[
\dot{\Theta} = \dot{\psi}' \equiv (\Theta')'(\psi) \dot{\psi},
\]

where $\dot{\psi}'$ denotes the derivative of $\dot{\psi} = f'$ with respect to $t$, so that $\dot{\psi}' = f''$, while $\Theta'$ denotes the derivative of $\Theta$ with respect to $x$.

For any Kähler class $\Omega_{a,b}$, the relative K-energy $E_{a,b}^G$ is determined, up to additive constant, via its differential on $\mathcal{M}_{\Omega_{a,b}}^{adm}$, defined by

\[
(dE_{a,b}^G)_{\omega_0}(f) = \int_M \left( \Pi_{g_0}^G(s_{g_0}) - s_{g_0} \right) f v_g
\]

\[
= 4\pi^2 \ell \int_{-\infty}^{+\infty} \left( \Pi_{g_0}^G(s_{g_0}) - s_{g_0} \right) f \psi' dt.
\]

By putting together (10.4.9) and (10.8.1), we get

\[
s_{g_0}^G := s_{g_0} - \Pi_{g_0}^G(s_{g_0}) = \frac{P_{\Omega_{a,b}}(\psi) - \psi \Theta(\psi)''}{\psi},
\]
where the second derivative is relative to $\psi$. We then rewrite (10.9.3) as

$$
(dE_{\Omega_{a,b}}^{G})_{\omega_\psi}(f) = 4\pi^2 \ell \int_{-\infty}^{+\infty} (\psi \Theta(\psi) - P_{\Omega_{a,b}}(\psi))'' f \psi' dt
$$

(10.9.5)

$$
d = 4\pi^2 \ell \int_{a}^{b} (x \Theta(x) - P_{\Omega_{a,b}}(x))'' f dx,
$$

where $(x \Theta(x) - P_{\Omega_{a,b}}(x))''$ denotes the second derivative with respect to $x$. By integrating by parts two times and by observing that the function $x\Theta(x) - P_{\Omega_{a,b}}(x)$ and its first derivative (with respect to $x$) both vanish at $x = a$ and $x = b$, we get

$$
(\Omega_{a,b}) dE_{\Omega_{a,b}}^{G}_{\omega_\psi}(f) = 4\pi^2 \ell \int_{a}^{b} (x \Theta(x) - P_{\Omega_{a,b}}(x)) \frac{d^2 f}{dx^2} dx.
$$

(10.9.6)

In this expression, $f = f(t)$ is regarded as a function of $x$, when $t$ is viewed as a function of $x$ whose first derivative is equal to $\frac{1}{\Theta(x)}$, cf. Proposition 10.3.4. We then have: $\frac{df}{dx} = \frac{\dot{\psi}}{\Theta}$ and $\frac{d^2 f}{dx^2} = \frac{\ddot{\psi}}{\Theta^2} - \frac{\dot{\psi} \dot{\Theta}}{\Theta^2}$, where, again, the derivatives are relative to the natural variable: $t$ for $\psi$ and $x$ for $\Theta$. From (10.9.2), we then obtain

$$
\frac{d^2 f}{dx^2} = \frac{\dot{\Theta}}{\Theta^2}.
$$

(10.9.7)

By reporting this expression in (10.9.6) we get:

$$
(\Omega_{a,b}) dE_{\Omega_{a,b}}^{G}_{\omega_\psi}(f) = 4\pi^2 \ell \int_{a}^{b} \frac{(x \Theta(x) - P_{\Omega_{a,b}}(x)\dot{\Theta}(x)}{\Theta^2(x)} dx,
$$

which is clearly the first variation of the rhs of (10.9.1).

Proposition 10.9.1 has the following two immediate consequences:

**Corollary 10.9.1.** The Calabi polynomial $P_{\Omega_{a,b}}$ is positive on $(a, b)$ if and only if $\mathcal{M}_{\Omega_{a,b}}^{adm}$ contains an extremal metric. This is then an absolute minimum of the restriction of $E_{\Omega_{a,b}}^{G}$ to $\mathcal{M}_{\Omega_{a,b}}^{adm}$.

**Proof.** When the restriction of $E_{\Omega_{a,b}}^{G}$ to $\mathcal{M}_{\Omega_{a,b}}^{G,adm}$ is viewed as a function defined on $\mathcal{A}_{a,b}$, we get from (10.9.1) the following expression of its first derivative at any element $\Theta$ of $\mathcal{A}_{a,b}$ along any variations $\dot{\Theta}$:

$$
(\Omega_{a,b}) (dE_{\Omega_{a,b}}^{G})_{\Theta}(\dot{\Theta}) = 4\pi^2 \ell \int_{a}^{b} \left( x - \frac{P_{\Omega_{a,b}}(x)}{\Theta(x)} \right) \frac{\dot{\Theta}(x)}{\Theta(x)} dx.
$$

(10.9.9)

It follows that $\Theta$ is critical if and only if $\Theta(x) = \frac{P_{\Omega_{a,b}}(x)}{x}$. This is only possible if $P_{\Omega_{a,b}}$ is positive on $(a, b)$, as $\Theta$ belongs to $\mathcal{A}_{a,b}$. Conversely, if $P_{\Omega_{a,b}}$ is positive on $(a, b)$ then $\Theta(x) = \frac{P_{\Omega_{a,b}}(x)}{x}$ is well defined as an element of $\mathcal{A}_{a,b}$ and the corresponding admissible Kähler metric is extremal, as $\Theta(\psi) = \psi'(t)$, whereas admissible extremal Kähler metrics $g_\psi$ are characterized by the equation $\psi \psi' = P_{\Omega_{a,b}}(\psi)$; this means that this metric is critical not only for the restriction of $E_{\Omega_{a,b}}^{G}$ to $\mathcal{M}_{\Omega_{a,b}}^{G,adm}$ but also for $E_{\Omega_{a,b}}^{G}$ defined on the whole space $\mathcal{M}_{\Omega_{a,b}}^{G}$. This fits with the already observed fact that no admissible extremal
Kähler metric exists in $\mathcal{M}_{\Theta_{a,b}}$ if the corresponding Calabi polynomial $P_{\Theta_{a,b}}$ has zeros on the open interval $(a, b)$. On the other hand, if $P_{\Theta_{a,b}}$ is positive on $(a, b)$, then $\Theta_0(x) := \frac{P_{\Theta_{a,b}}(x)}{x}$ belongs to $\mathcal{A}_{a,b}$, hence determines an admissible extremal metric, and we then have

$$E_{\Omega_{a,b}}^G(\Theta_0) = 4\pi^2\ell \int_a^b x(1 + \log \frac{P_{\Theta_{a,b}}(x)}{x}) \, dx.$$  

(10.9.10)

It is easy to check that for any other $\Theta$ in $\mathcal{A}_{a,b}$, we have

$$E_{\Omega_{a,b}}^G(\Theta) - E_{\Omega_{a,b}}^G(\Theta_0) \geq 0,$$

with equality if and only if $\Theta = \Theta_0$. Indeed, $E_{\Omega_{a,b}}^G(\Theta) - E_{\Omega_{a,b}}^G(\Theta_0)$ can be written as

$$E_{\Omega_{a,b}}^G(\Theta) - E_{\Omega_{a,b}}^G(\Theta_0) = 4\pi^2\ell \int_a^b x \frac{P_{\Theta_{a,b}}(x)}{x \Theta(x)} - 1 - \log \frac{P_{\Theta_{a,b}}(x)}{x \Theta(x)} \, dx,$$

where the integrand is clearly non negative and identically zero if and only if $\frac{P_{\Theta_{a,b}}(x)}{x \Theta(x)} = 1$ (the function $h(x) := x - 1 - \log x$ is non negative on the positive half line $(0, +\infty)$ and $h(x) = 0$ if and only if $x = 1$). It follows that if $\mathcal{M}_{\Theta_{a,b}}^{G,\text{adm}}$ contains an extremal metric, say $\omega_0$, then this metric is a global minimum for the relative K-energy $E_{\Omega_{a,b}}^G$ on $\mathcal{M}_{\Theta_{a,b}}^{G,\text{adm}}$.

**Remark 10.9.1.** The last statement of Corollary 10.9.1 fits with Theorem 4.15.2, which says that $\omega_0$ is actually a global minimum of $E_{\Omega_{a,b}}^G$ on the whole space $\mathcal{M}_{\Theta_{a,b}}^G$.

The second corollary is a particular case of [6, Corollary 2]:

**Corollary 10.9.2.** If $P_{\Theta_{a,b}}$ is negative on a non-empty open subinterval $(a', b')$ of $(a, b)$, then $E_{\Omega_{a,b}}^G$ is unbounded from below on $\mathcal{M}_{\Theta_{a,b}}^G$.

**Proof.** We actually prove the stronger result that $E_{\Omega_{a,b}}^G$ is unbounded from below on $\mathcal{M}_{\Theta_{a,b}}^{G,\text{adm}}$. The argument reproduces the one in [6, Corollary 2]. Choose any $\Theta_0$ in $\mathcal{A}_{a,b}$ and, for any non-negative real number $s$, define $\Theta_s := \frac{\Theta_0}{1 + s \rho \Theta_0}$, where $\rho$ stands for any non-negative function defined on $(a, b)$, with support in $(a', b')$; then, $\Theta_s$ belongs to $\mathcal{A}_{a,b}$ for any $s \geq 0$, and we have

$$E_{\Omega_{a,b}}^G(\Theta_s) = E_{\Omega_{a,b}}^G(\Theta_0) + 4\pi^2\ell \int_{a'}^{b'} s \rho(x) P_{\Theta_{a,b}}(x) \, dx$$

$$- 4\pi^2\ell \int_a^b x \log (1 + s \rho(x) \Theta_0(x)) \, dx,$$

(10.9.13)

where the rhs clearly tends to $-\infty$ when $s$ tends to $+\infty$. □

Corollary 10.9.2 together with Theorem 4.15.2 imply the following theorem, which provides a full answer to the question asked in Remark 10.4.1.

**Theorem 10.9.1 ([6] Corollary 2).** Let $M = \mathbb{P}(1 \oplus L)$ be a Hirzebruch-like ruled surface of any genus $g$ and any degree $\ell$. Let $\Omega_{a,b}$ be any Kähler class on $M$ and $P_{\Theta_{a,b}}$ be the corresponding Calabi polynomial. Then $M$
admits an extremal Kähler metric of Kähler class $\Omega_{a,b}$ if and only if $P_{\Omega_{a,b}}$ is positive on the open interval $(a, b)$.

**Proof.** We already know that $M$ admits an admissible extremal Kähler metric within $\Omega_{a,b}$ whenever the extremal polynomial $P_{\Omega_{a,b}}$ is positive on $(a, b)$ and that this happens for all admissible ruled surfaces of genus 0 or of genus 1 and for all admissible ruled surfaces of genus $g > 1$ and degree $\ell$, provided $x = \frac{b}{a} < x_\kappa$, cf. Theorem 10.4.1. If $g > 1$ and $x > x_\kappa$, $P_{\Omega_{a,b}}$ is negative on some non-empty open subinterval of $(a, b)$ and the relative K-energy is then not bounded from below on $M_{\Omega_{a,b}}$, as we just checked: from Theorem 4.15.2 we then infer that $M$ admits no extremal metric of Kähler class $\Omega_{a,b}$. It remains the case when $g > 1$ and $x = x_\kappa$; in this case, we derive the non-existence of any extremal metric from the fact that the extremal Kähler cone is open, cf. Chapter 5: if there were an extremal Kähler metric in a Kähler class $\Omega_{a,b}$ with $\frac{b}{a} = x_\kappa$ there would exist extremal Kähler metrics in all Kähler classes $\Omega_{a,b}$ with $\frac{b}{a}$ in $(x_\kappa - \varepsilon, x_\kappa + \varepsilon)$ for some $\varepsilon > 0$; by the above, this cannot be. \qed
Bochner tensor and Apte formulae

A.1. Curvature decomposition

The curvature, \( R \), of a general \( n \)-dimensional riemannian manifold \((M, g)\) is defined by (1.18.1), where \( R \) appears as a 2-forms with values in the bundle, \( A_M \), of skew-symmetric endomorphisms of \( TM \). We make it into a 4-linear form, still denoted by \( R \), by putting:

\[
R_{X_1,X_2,X_3,X_4} = g(R_{X_1,X_2,X_3,X_4}).
\]

Then, \( R \) satisfies the following three identities:

1. \( R_{X_1,X_2,X_3,X_4} + R_{X_2,X_1,X_3,X_4} = 0 \),
2. \( R_{X_1,X_2,X_3,X_4} + R_{X_1,X_2,X_4,X_3} = 0 \),
3. \( R_{X_1,X_2,X_3,X_4} + R_{X_2,X_3,X_1,X_4} + R_{X_3,X_1,X_2,X_4} = 0 \),

for all vector fields \( X_1, X_2, X_3, X_4 \). The first identity holds for the curvature of any sort of linear connection. The second identity holds for the curvature of any \( g \)-metrical linear connection. The third identity is the (algebraic) Bianchi identity and holds for the curvature of any torsion-free linear connection, cf. Section 1.18. These identities imply the following fourth one:

\[
R_{X_1,X_2,X_3,X_4} = R_{X_3,X_4,X_1,X_2},
\]

which can be considered as a weak version of the (algebraic) Bianchi identity.

The riemannian curvature \( R \) is then a section of a vector bundle whose fiber is the subvector space, \( \mathcal{R}(n) \), of elements of \( \otimes^4 V^* \) which satisfy the above identities (where \( V \) stands for \( \mathbb{R}^n \)). This vector space is called the space of abstract riemannian curvature tensors. It can be viewed as a representation space of the full linear group \( GL(n, \mathbb{R}) \) or as a representation space of the orthogonal group \( O(n) \). As a representation space of \( GL(n, \mathbb{R}) \), \( \mathcal{R}(n) \) is irreducible, meaning that the riemannian curvature admits no additional general symmetries. As a representation space of \( O(n) \), \( \mathcal{R}(n) \) is still irreducible if \( n = 2 \), splits into two irreducible components if \( n = 3 \), into three irreducible components if \( n > 3 \). If \( n = 4 \), \( \mathcal{R}(n) \) splits into three irreducible components under the action of \( O(n) \), but into four irreducible components under the action of the special orthogonal group \( SO(4) \): for \( n > 4 \), there is no difference between the decomposition into irreducible components under \( O(n) \) and \( SO(n) \).

By (A.1.1), \( \mathcal{R}(n) \) is a subspace of the symmetric tensor product \( \Lambda^2(n) \odot \Lambda^2(n) \), where \( \Lambda^2(n) \) stands for \( \Lambda^2 V^* \). The restriction of the Bianchi projection to \( \Lambda^2(n) \odot \Lambda^2(n) \) takes values in the space \( \Lambda^4(n) = \Lambda^4 V^* \) and reads:

\[
\kappa(\phi \otimes \phi) = \frac{1}{6} \phi \wedge \phi,
\]

for each \( \phi \) in \( \Lambda^2(n) \); \( \kappa \) is a \( GL(n, \mathbb{R}) \)-equivariant map from \( \Lambda^2(n) \odot \Lambda^2(n) \) onto \( \Lambda^4(n) \), whose kernel is \( \mathcal{R}(n) \).
We denote by $\text{Sym}(n)$ the space of symmetric elements of $V^* \otimes V^*$, by $\text{Sym}_0(n)$ the subspace of trace-free elements of $\text{Sym}(n)$.

The Ricci trace is the $O(n)$-equivariant map, $c$, from $\mathcal{R}(n)$ to $\text{Sym}(n)$ defined by:

$$c(R)_{X,Y} = \sum_{j=1}^{n} R_{X,e_j,Y,e_j},$$

for any orthonormal basis $e_1, \ldots, e_n$ of $V$.

For $n \geq 3$, this map is onto (an isomorphism if $n = 3$). For $n > 3$, the kernel of $c$ is denoted by $\mathcal{W}(n)$ and called the space of (abstract) Weyl tensors. The orthogonal complement of $\mathcal{W}(n)$ in $\mathcal{R}(n)$ coincides with the image of the (metric) adjoint $c^*$ of $c$. In order to give an explicit expression of $c^*$, it is convenient to consider elements of $\mathcal{R}(n)$ as (symmetric) maps from $\Lambda^2(n)$ to itself, via the metric $g$. In the same way, elements of $\Lambda^2(n)$ and of $\text{Sym}(n)$ may be viewed as endomorphisms of $V$, respectively skew-symmetric and symmetric. We then have

$$c^*(S) : \phi \mapsto \{S, \phi\} := S \circ \phi + \phi \circ S,$$

for any $S$ in $\text{Sym}(n)$ and any $\phi$ in $\Lambda^2(n)$.

It is easily checked that

$$cc^*(S) = \text{tr} SI + (n - 2) S,$$

where $\text{tr} S$ denotes the trace of $S$ and $I$ stands for the identity of $V$, viewed as an element of $\text{Sym}(n)$. The equation $cc^*(S) = r$ is thus solved by

$$S = \frac{s}{2n(n-1)} I + \frac{1}{(n-2)} r_0,$$

where $s$ denotes the trace of $r$ and $r_0$ stands for the trace-free part of $r$.

We thus get the well-known decomposition of any element $R$ of $\mathcal{R}(n)$, hence of the curvature $R$ of any $n$-dimensional riemannian manifold $(M,g)$ (viewed as a symmetric endomorphism of $\Lambda^2M$):

$$R = \frac{s}{n(n-1)} I\big|_{\Lambda^2M} + \frac{1}{(n-2)} \{r_0, \cdot\} + W,$$

where: $r = c(R)$, $s = \text{tr}(r)$ and $W$, the Weyl tensor, is defined as the orthogonal projection of $R$ into the kernel of $c$.

As a $AM$-valued 2-form on $M$, the Weyl tensor $W$ only depends upon the conformal class of $g$. If the latter is flat, i.e. if $g$ can be locally made into a flat metric by conformal change, $W$ is then identically zero. If $n > 3$, the converse is true: $W \equiv 0$ implies that the conformal class of $g$ is flat.

If $n = 4$, then the orthogonal Lie algebra $\mathfrak{so}(4)$ splits, as a Lie algebra, as the sum of two copies of the orthogonal Lie algebra $\mathfrak{so}(3)$. Accordingly $\Lambda^2(4) \cong \mathfrak{so}(4)$ splits as the orthogonal sum $\Lambda^+ \oplus \Lambda^-$, where each $\Lambda^\pm$ is isomorphic to $\mathfrak{so}(3)$ (the choice of $\pm$ depends on the choice of an orientation of $V = \mathbb{R}^4$). If $*$ denotes the corresponding Hodge operator, then $*$ acts as a $SO(4)$-equivariant involution on $\Lambda^2(4)$ and $\Lambda^\pm$ is the eigenspace of $*$.
A.2. THE BOCHNER TENSOR

With respect to the eigenvalue ±1. Each Λ± is an irreducible representation space under the (adjoint) action of SO(4). As a SO(4)-representation space, Sym0(4) is isomorphic to the tensor product Λ+ ⊗ Λ−: the isomorphism Λ+ ⊗ Λ− → Sym0(4) is defined by ψ+ ⊗ ψ− → ψ+ ◦ ψ−, for any ψ± in Λ±, both viewed as (commuting) skew-symmetric endomorphisms of V. It follows that the r0-part of R in (A.1.7) exchanges Λ+ and Λ−. In other words, the r0-part of R anti-commutes with ∗. On the other hand, the Weyl tensor W commutes with ∗, hence splits as W = W+ ⊕ W−, where W+ resp. W−, acts trivially on Λ− resp. Λ+. Both W+ and W− are conformally invariant. The algebraic splitting Λ2(4) = Λ+ ⊕ Λ− implies a similar splitting Λ2M = Λ+M ⊕ Λ−M of the bundle of (real) 2-forms into the orthogonal direct sum of the bundle Λ+M of selfdual 2-forms, on which the Hodge operator acts as the identity, and the bundle Λ−M of antiselfdual 2-forms, on which the Hodge operator acts as minus the identity. With respect to this decomposition, the curvature can then be pictured as follows

(A.1.8) \[ R = \begin{pmatrix} W^+ + \frac{s}{12} I_{Λ^+M} & \frac{1}{2} \{r_0, \cdot \} \\ \frac{1}{2} \{r_0, \cdot \} & W^- + \frac{s}{12} I_{Λ^-M} \end{pmatrix} \]

where \( I_{Λ±M} \) stands for the identity of \( Λ±M \) and where \( W± \), which acts trivially on \( Λ±M \), is here considered as a (trace-free, symmetric) endomorphisms of \( Λ±M \), cf. [179].

A.2. The Bochner tensor

We now consider the case when \( n = 2m \) the metric g is kählerian with respect to some complex structure J. Then, \( Λ2M \) splits as

(A.2.1) \[ Λ^J+M ⊕ Λ^J−M, \]

where \( Λ^J±M \) denotes the bundle of ±-invariant (real) 2-forms on M under the natural action of J on \( Λ2M \) (which acts as an involution) and the curvature R, viewed as a symmetric endomorphism of \( Λ2M \), acts trivially on the factor \( Λ^J−M \). From an algebraic point of view, the decomposition (A.2.1) corresponds to the decomposition of \( U(m) \)-representation spaces

(A.2.2) \[ so(2m) = u(m) ⊕ u(m)\perp \]

where \( u(m) \) denotes the unitary group of order m, \( u(m) = Λ^{1,1}(m) \) its Lie algebra, \( u(m)\perp \) the orthogonal of \( u(m) \) in \( so(2m) = Λ^2(2m) \). The space \( K(m) \) of abstract kählerian curvature tensors is then defined as the space of elements of \( R(2m) \) which act trivially on the factor \( u(m)\perp \) or, equivalently, as the kernel of the restriction of the Bianchi projector \( κ \) to \( Λ^{1,1}(m) \bigcirc Λ^{1,1}(m) \), which takes its values in the space \( Λ^{2,2}(m) \): \( K(m) \) is then a subspace of \( R(2m) \) and both are now regarded as representation spaces of \( U(m) \). A first difference with the preceding situation is that neither \( Λ^{1,1}(m) \) nor \( Λ^{2,2}(m) \) are irreducible under the action of \( U(m) \); we have instead: \( Λ^{1,1}(m) = Λ^{0,1}(m) \bigoplus \mathbb{R} \) and \( Λ^{2,2}(m) = Λ^{2,0}(m) \bigoplus Λ^{1,1}(m) \bigoplus \mathbb{R} \), where \( \mathbb{R} \) stands for the trivial representation space.

If R belongs to \( R(m) \) the components of R which appear in (A.1.7) don’t belong to \( K(m) \); in particular, the scalar part \( \frac{s}{m(n-1)} I_{Λ^2(2m)} \) definitely does not belong to \( K(m) \) if \( s \neq 0 \) (see however section A.4 below for the case when \( m = 2 \)).
The restriction of the Ricci trace $c$ to $\mathcal{K}(m)$ will be denoted by $c_K$. It is easily checked that $c_K$ takes its values in the subspace $\text{Sym}^{1,1}(m)$ of $J$-invariant elements of $\text{Sym}(m)$. Moreover, as a representation space of $U(m)$, $\text{Sym}^{1,1}(m)$ is the same as $\Lambda^{1,1}(m)$, via the map: $S \to \sigma := J \circ S$ (beware, however, that this “identification” does not preserve the natural metrics of the two spaces; more precisely, according to our special convention for the norm of exterior forms, see Section 1.3, we have that $|J \circ S|^2 = \frac{1}{2}|S|^2$). Then, a simple application of the algebraic Bianchi identity shows that
\begin{equation}
(A.2.3) \quad c(R) = c_K(R) = R(\omega),
\end{equation}
for any $R$ in $\mathcal{K}(m)$.

The map $c_K$ is a $U(m)$-equivariant map from $\mathcal{K}(m)$ onto $\text{Sym}^{1,1}(m)$. It is an isomorphism if $m = 1$.

We henceforth assume that $m \geq 2$.

The kernel of $c_K$, i.e. the intersection $\mathcal{W}(2m) \cap \mathcal{K}(m)$, is called the space of abstract Bochner tensors, denoted by $\mathcal{W}^K(m)$.

The orthogonal complement of $\mathcal{W}^K(m)$ in $\mathcal{K}(m)$ is the image of the metric adjoint $c^*_K$ of $c_K$, from $\text{Sym}^{1,1}(m)$ to $\mathcal{K}(m)$ (for the natural metric of $\text{Sym}^{1,1}(m)$). It can be checked that $c^*_K$ has the following expression:
\begin{equation}
(A.2.4) \quad c^*_K(S) : \phi \to \{S, \phi^{1,1}\} + (\phi, \omega) \sigma + (\sigma, \phi) \omega,
\end{equation}
for each $\phi$ in $\Lambda^2(2m)$ and each $S$ in $\text{Sym}^{1,1}(m)$ (here, $\sigma$ denotes the corresponding $J$-invariant 2-form; the scalar product is the scalar product in $\Lambda^{1,1}(m)$).

The equation $c_K c^*_K(S_K) = r$ is now solved by
\begin{equation}
(A.2.5) \quad S_K = \frac{2}{(m+2)} r + \frac{s}{2m(m+1)} g.
\end{equation}

We thus get the following decomposition into $U(m)$-irreducible components of $\mathcal{K}(m)$, hence also the decomposition of the curvature $R$ of any Kähler metric:
\begin{equation}
(A.2.6) \quad R = \frac{s}{2m(m+1)} (\Pi^{1,1} + \omega \otimes \omega)
\end{equation}
\begin{equation*}
+ \frac{1}{(m+2)} (\{r_0, \cdot\} + \omega \otimes \rho_0 + \rho_0 \otimes \omega)
\end{equation*}
\begin{equation*}
+ W^K,
\end{equation*}
where $\Pi^{1,1}$ stands for the orthogonal projection from $\Lambda^2(2m)$ onto $\Lambda^{1,1}(m)$, $\rho_0$ denotes the primitive part of the Ricci form $\rho$, whereas $W^K$ denotes the Bochner tensor, defined as the orthogonal projection of $R$ into the space of Bochner tensors.

We observe that the first line, namely the (kählerian) scalar part of $R$, is the curvature of any $2m$-dimensional Kähler manifold of constant holomorphic sectional curvature $c = \frac{s}{m(m+1)}$.

Again, the case when $m = 2$ deserves a special mention. Then, the (riemannian) $r_0$-part of $R$ appearing in (A.1.7) does belong to $\mathcal{K}(m)$ for each $R$ in $\mathcal{K}(m)$: indeed, any $J$-anti-invariant element $\phi$ of $\Lambda^2(m)$ belongs to $\Lambda^+$; it is then sent to $\Lambda^-$ by the $r_0$-part of $R$; on the other hand, $\{r_0, \phi\}$ is certainly $J$-anti-invariant again, as $r_0$ is $J$-invariant, hence belongs to $\Lambda^+$;
we conclude that $\phi$ is killed by the riemannian $r_0$-part of $R$; this proves that the riemannian $r_0$-part of $R$ already belongs to $K(m)$. On the other hand, $W^-$ also kills the $J$-anti-invariant part of $\Lambda^2(m)$, which is included in $\Lambda^+$; it follows that $W^-$ also belongs to $K(m)$. Finally, it is easily checked that the (riemannian) scalar part of $R$ put together with $W^+$ is equal to the kählerian scalar part of $R$. It follows that the riemannian $r_0$-part of $R$ coincides with its kählerian $r_0$-part, whereas the Bochner tensor $W^K$ coincides with $W^-$. As in [28], Chapter 2, we get a coarser decomposition of a kählerian curvature tensor $R$ by just considering $R$ as an element of $\Lambda^1_0 \otimes \Lambda^1_0 \otimes \Lambda^1_0$ and by decomposing $\Lambda^1_0$ as $\Lambda^1_0 = \Lambda^1_0 \otimes \mathbb{R} \omega$. Then, $R$ induces a symmetric endomorphism of $\Lambda^1_0 \otimes \Lambda^1_0$, denoted by $B$. If we denote by $B_0$ the trace-free part of $B$, we end up with the following decomposition $R$:

$$R = \frac{s}{2m(m+1)} (\Pi^{1,1} + \omega \otimes \omega) + \frac{1}{m} (\rho_0 \otimes \omega + \omega \otimes \rho_0) + B_0,$$

or, by introducing an “orthonormal” (local) frame of $\Lambda^1_0 M$, so that $(\omega_r, \omega_s) = m \delta_{rs}$,

$$R = \frac{s}{2m^2} \omega \otimes \omega + \frac{1}{m} (\rho_0 \otimes \omega + \omega \otimes \rho_0) + \frac{1}{m} \sum_{r=1}^{m^2-1} \frac{s}{2m(m+1)} + \lambda_r \omega_r \otimes \omega_r,$$

where the $\lambda_r$ denote the eigenvalues of $B_0$.

The tensor $B_0$ coincides with $W^-$, hence with $W^K$, if $m = 2$, but in general we have that:

$$B_0 = W^K + \frac{2}{m(m+2)} (\rho_0 \otimes \omega + \omega \otimes \rho_0) - \frac{1}{(m+2)} \{r_0, \cdot\}.$$

It follows that

$$|B_0|^2 = |W^K|^2 + \frac{2(m-2)}{m(m+2)} |\rho_0|^2.$$

A.3. Apte formulae

Formula (A.2.8) turns out to be very convenient for establishing the following two formulae, known as Apte formulae [12], [25, Chapitre XII] [28, Chapter 2]:

\begin{align*}
\text{apte1} & \quad (c_2^2 \cup \Omega^{m-2})[M] = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{4m^2} - \frac{|\rho_0|^2}{m(m-1)} \right) \omega^m, \\
\text{apte2} & \quad (c_2 \cup \Omega^{m-2})[M] = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{4m(m+1)} - \frac{2 |\rho_0|^2}{(m-1)(m+2)} + \frac{|W^K|^2}{m(m-1)} \right) \omega^m,
\end{align*}
where, we recall, $\Omega = [\omega]$ denotes the de Rham class of $\omega$, and $c_1, c_2$ denote the first and second Chern class of $(M, J)$, in $H^2_{dR}(M, \mathbb{R})$ and $H^4_{dR}(M, \mathbb{R})$ respectively.

Before proving (A.3.1) and (A.3.2), we recall the following facts. For any $\phi$ in $\Lambda^{1,1}(m)$, denote by $\Phi = -J \circ \phi$ the corresponding hermitian operator, acting on $V = \mathbb{R}^{2m}$ viewed as a $m$-dimensional complex vector space. As in Section 1.1.2, denote by $P(t) = \sum_{k=1}^{m} (-1)^r \sigma_r(\phi) t^{m-r}$ the characteristic polynomial of $\Phi$. Then, each $\sigma_r$ is a homogeneous $U(m)$-invariant polynomial of degree $r$ on the Lie algebra $\Lambda^{1,1}(m) = u(m)$ and, by definition, the corresponding characteristic class is the $r$-th Chern class. This means that for any Kähler structure, of curvature $R$, the $r$-th Chern class $c_r$ of $(M, J)$, viewed as an element of $H^{2r}(M, \mathbb{R})$, is represented by the (closed) $2r$-class $c_r(R)$ defined by

$$c_r(R) = \frac{1}{(2\pi)^r} \sigma_r(R \wedge \ldots \wedge R),$$

($r$ times), with the following significance: $R$ is a $\Lambda^{1,1}M$-valued 2-form; the formal exterior product $R \wedge \ldots \wedge R$ has to be understood as a 2-form with values in the tensor product $\Lambda^{1,1}M \otimes \ldots \otimes \Lambda^{1,1}M$ ($r$ times); we finally contract by $\sigma_r$, viewed as a $r$-linear form on each fiber of $\Lambda^{1,1}M$, in order to get a real 2r-form.

The differential Bianchi identity

$$d^D R = 0,$$

implies that $\sigma_r(R)$ is closed and that its de Rham class is independent of the chosen Kähler metric. The factor $\frac{1}{(2\pi)^r}$ ensures that the corresponding element of $H^{2r}(M, \mathbb{R})$ is integral, i.e. belongs to the image of $H^{2r}(M, \mathbb{Z})$.

By using (1.12.2)-(1.12.4)-(1.12.5) of Section 1.12, we are now ready to prove (A.3.1)-(A.3.2).

The LHS of (A.3.1) is equal to $\frac{1}{4\pi^2} \int_M \rho \wedge \rho \wedge \omega^{m-2}$. We then have:

$$\frac{1}{4\pi^2} \int_M \rho \wedge \rho \wedge \omega^{m-2} = \frac{1}{4\pi^2} (m-2)! \int_M \Lambda^2(\rho \wedge \rho) \, v_g$$

$$= \frac{1}{4\pi^2} \frac{1}{m(m-1)} \int_M \Lambda^2(\rho \wedge \rho) \omega^m$$

$$= \frac{1}{4\pi^2} \frac{1}{m(m-1)} \int_M (\frac{s^2}{4} - |\rho|^2) \omega^m$$

$$= \frac{1}{4\pi^2} \frac{1}{m(m-1)} \int_M (\frac{(m-1)s^2}{4m} - |\rho|^2) \omega^m.$$  

This proves (A.3.1).

The LHS of (A.3.2) is equal to $\frac{1}{4\pi^2} \int_M \sigma_2(R \wedge R) \wedge \omega^{m-2}$. We then have:

$$\frac{1}{4\pi^2} \int_M \sigma_2(R \wedge R) \wedge \omega^{m-2} = \frac{1}{4\pi^2} (m-2)! \int_M \Lambda^2(\sigma_2(R \wedge R) \, v_g$$

$$= \frac{1}{4\pi^2} \frac{1}{m(m-1)} \int_M \Lambda^2(\sigma_2(R \wedge R) \omega^m.$$
By using (1.12.4) and (A.2.8), we get
\[
\Lambda^2(\sigma_2(R \wedge R)) = \frac{s^2}{m^2} \sigma_2(\omega, \omega) \sigma_2(\omega, \omega)
\]
\[
+ \frac{8}{m^2} \sigma_2(\omega, \omega) \sigma_2(\rho_0, \rho_0)
\]
\[
+ \frac{s^2}{m^4(m+1)^2} \sum_{r=1}^{m^2-1} \sigma_2(\omega_r, \omega_r) \sigma_2(\omega_r, \omega_r)
\]
\[
+ \frac{4}{m^2} \sum_{r=1}^{m^2-1} \sigma_2(\omega_r, \omega_r) \sigma_2(\omega_r, \omega_r).
\]

By using (1.12.5), we get
\[
(A.3.5) \quad \Lambda^2(\sigma_2(R \wedge R)) = \frac{(m-1)}{4(m+1)} s^2 - \frac{2(m-1)}{m} |\rho_0|^2 + |B_0|^2.
\]

By (A.2.10), this gives (A.3.2).

**Proposition A.3.1** (S.-T. Yau [197]). A compact kählerian manifold with \(c_1(M) = 0\) and \(c_2(M) = 0\) admits a (unique) flat Kähler metric in each Kähler class.

**Proof.** Since \(c_1(M) = 0\), it follows from Calabi-Yau theorem that each Kähler class contains a unique Ricci flat Kähler metric. It then follows from (A.3.1)-(A.3.2) that such a metric \(g\) also have: \(s_g = 0\) and \(W^K = 0\), hence is flat. \(\square\)

**A.4. The four-dimensional case**

One specific feature of the curvature of a Kähler manifold of (real) dimension 4 is that the term \(\{r_0, \cdot\}\) in the decomposition (A.1.7) is also of Kähler type, i.e. acts trivially on \(\Lambda^{J,-} M\). This is because, \(\Lambda^{J,-} M\) is included in \(\Lambda^+ M\), so that \(\{r_0, \psi\}\) is antiselfdual for any \(\psi\) in \(\Lambda^{J,-} M\); on the other hand, \(\{r_0, \psi\}\) certainly anti-commutes with \(J\), so that \(\{r_0, \psi\}\) is selfdual; it follows that \(\{r_0, \psi\} = 0\) for each \(\psi\) in \(\Lambda^{J,-} M\).

On the other hand, since \(R\) acts trivially on \(\Lambda^{J,-} M\) the selfdual Weyl tensor \(W^+\) in the Singer-Thorpe decomposition (A.1.8) is reduced to

\[
W^+ = \left( \begin{array}{ccc} \frac{s}{12} & 0 & 0 \\ 0 & -\frac{s}{12} & 0 \\ 0 & 0 & -\frac{s}{12} \end{array} \right)
\]

with respect to the decomposition \(\Lambda^+ M = \mathbb{R} \omega \oplus \Lambda^{J,-} M\) or else

\[
(A.4.2) \quad W^+ = -\frac{s}{12} \Pi^+ + \frac{s}{8} \omega \otimes \omega.
\]

In particular,

\[
(A.4.3) \quad |W^+|^2 = \frac{s^2}{24}.
\]

In general, the selfdual Weyl tensor \(W^+\) is not of Kähler type, neither the scalar part \(\frac{1}{12} I_{\Lambda^2 M}\) of \(R\) in (A.1.7), but the sum \(\frac{1}{12} I_{\Lambda^2 M} + W^+ = \frac{1}{12} (\Pi^{1,1} + \omega \otimes \omega)\)
\( \omega \otimes \omega \) certainly is and actually coincides with the scalar part of \( R \) in (A.2.6).

It follows that the latter can be written

\[
R = \frac{s}{12}(\Pi^{1,1} + \omega \otimes \omega)
\]

(A.4.4)

\[
+ \frac{1}{2}(\omega \otimes \rho_0 + \rho_0 \otimes \omega)
\]

\[
+ W^-. 
\]

In particular, we have that

(A.4.5) \( W^K = W^- \).

The Apte formulae then become

\[
c_1^2[M] = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{8} - |\rho_0|^2 \right) v_g,
\]

(A.4.6)

\[
c_2[M] = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{12} - |\rho_0|^2 + |W^-|^2 \right) v_g.
\]

We infer

(A.4.7) \( (c_1^2 - 2c_2)[M] = \frac{1}{4\pi^2}(\int_M \frac{s^2}{24} - |W^-|^2) v_g \).

It turns out that for any compact complex surface \((M, J)\) the two Chern numbers \( c_1^2[M] \) and \( c_2[M] \) only depends on the underlying oriented 4-manifold. More precisely, if \( \chi \) and \( \tau \) denote the Euler characteristic and the signature of \( M \), we have that

(A.4.8) \( \chi = c_2[M], \quad \tau = \frac{1}{3}(c_1^2 - 2c_2)[M] \).

**Remark A.4.1.** For any compact (oriented, connected) 4-dimensional riemannian manifold \((M, g)\), the *Gauss-Bonnet formula* gives the following general expression for the Euler characteristic:

\[
\chi = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} - \frac{|r_0|^2}{2} + |W|^2 \right) v_g,
\]

(A.4.9)

where \( r_0 \) denotes the trace-free part of the Ricci tensor \( r \), whereas, by the *Hirzebruch signature formula*, the signature is expressed as

\[
\tau = \frac{1}{12\pi^2} \int_M (|W^+|^2 - |W^-|^2) v_g.
\]

(A.4.10)

In the Kähler case, these expressions can be readily deduced from (A.4.6), (A.4.7), (A.4.8) and (A.4.3) (beware however that according to our general convention — cf. Section 1.3 — we have that \( |\rho_0|^2 = \frac{|r_0|^2}{2} \)) and the square norms \( |W|^2, |W^+|^2, |W^-|^2 \) which appear in (A.4.9)-(A.4.10) and in other parts of these notes may differ by a factor \( \frac{1}{4} \) from other conventions occurring in the literature).
APPENDIX B

The space of almost complex structures compatible with a symplectic form

B.1. The Cayley transformation

In this section, $E \cong \mathbb{R}^{2m}$ denotes a real vector space of even dimension $n = 2m$ and we fix a complex structure $J_0$ of $E$.

For any complex structure $J$ of $E$, we denote by $E^{1,0}_J$, resp. $E^{0,1}_J$, the subspace of elements of type $(1, 0)$, resp. $(0, 1)$, of $E^C := E \otimes \mathbb{C}$ with respect to $J$.

Then, $J$ is said to be commensurable to $J_0$ if the projection of $E^{0,1}_J$ on $E^{0,1}_{J_0}$ along $E^{1,0}_{J_0}$ is onto, hence an isomorphism (in particular $J_0$ is always commensurable to itself but never to its conjugate $-J_0$).

By definition, $J$ is commensurable to $J_0$ if and only if $E^{0,1}_J \cap E^{1,0}_{J_0} = \{0\}$, if and only if $E^{0,1}_J$ is the graph of a $\mathbb{C}$-linear homomorphism, $\mu$, from $E^{0,1}_{J_0}$ to $E^{1,0}_{J_0}$, i.e.

\[
E^{0,1}_J = \{ Z + \mu(Z) \mid Z \in E^{0,1}_{J_0} \}.
\]

The homomorphism $\mu$ is the restriction to $E^{0,1}_J$ of a unique real $\mathbb{C}$-linear endomorphism of $E^C$, still denoted by $\mu$ as well as its restriction to $E$; the latter is then a $\mathbb{R}$-linear endomorphism of $E$ which anticommutes with $J_0$: $\mu \circ J_0 = -J_0 \circ \mu$.

Conversely, any such $\mu$ determines a complex structure $J$ of $E$ commensurable to $J_0$ if and only if the space $E^{0,1}_J$ defined by (B.1.1) satisfies the condition $E^{0,1}_J \cap E^{1,0}_{J_0} = \{0\}$. This condition is satisfied if and only if $1 - \mu^2$ is one to one, or, equivalently, if and only if $1 + \mu$ and $1 - \mu$ are both invertible (as endomorphisms of $E$).

The type decomposition of any element $X$ of $E$ relatively to $J$ can be written as follows:

\[
X = Z + \mu(Z) + Z + \mu(Z),
\]

where $Z$ is a well-defined element of $E^{1,0}_{J_0}$; then, $X_1 = Z + \overline{Z}$ is a well-defined element of $E$, with

\[
X = X_1 + \mu(X_1),
\]

and

\[
JX = J_0X_1 - J_0\mu(X_1).
\]

We infer that

\[
J = J_0(1 - \mu)(1 + \mu)^{-1} = (1 + \mu) J_0 (1 + \mu)^{-1}.
\]
It follows that \( J + J_0 = 2J_0(1 + \mu)^{-1} \) is invertible and that
\[
\mu = (J + J_0)^{-1}(J_0 - J) = (J - J_0)(J + J_0)^{-1}.
\]

Conversely, for any complex structure \( J \) of \( E \) such that \( J + J_0 \) is invertible, the endomorphism \( \mu \) of \( E \) defined by (B.1.6) is \( J_0 \)-antilinéaire (direct consequence of the easy identity \( J_0(J + J_0)^{-1} = (J + J_0)^{-1}J \)); moreover, \( 1 + \mu = 2J(J + J_0)^{-1} = 2(J + J_0)^{-1}J_0 \) and \( 1 - \mu = 2J_0(J + J_0)^{-1} = 2(J + J_0)^{-1}J \) are both invertible.

The above discussion can be summarized as follows:

**Proposition B.1.1.** (i) A complex structure \( J \) of \( E \) is commensurable with \( J_0 \) if and only if \( J + J_0 \) is invertible.

(ii) The Cayley correspondence \( J \leftrightarrow \mu \) given by (B.1.5)-(B.1.6) then establishes a natural identification of the space of \( J_0 \)-commensurable complex structures of \( E \) with the space of \( J_0 \)-antilinear endomorphisms of \( E \) such that \((1 \pm \mu)\) are invertible.

In the sequel, \( \mu \) will be referred to as the Cayley transform of \( J \).

The differential of the map \( \mu \to J \) is easily checked to be given by
\[
\mu \to \dot{J} = [\dot{\mu}(1 + \mu)^{-1}, J].
\]

We infer
\[
J_0\dot{\mu} \to J\dot{J}.
\]

Indeed, we have that \([\dot{\mu}(1 + \mu)^{-1}, J] = \dot{\mu}(1 + \mu)^{-1}J - J\dot{\mu}(1 + \mu)^{-1}\), whereas
\[
[J_0\dot{\mu}(1 + \mu)^{-1}, J] = -[\dot{\mu}J_0(1 + \mu)^{-1}, J] = -[\dot{\mu}(1 + \mu)^{-1}J, J] = \dot{\mu}(1 + \mu)^{-1} + J\dot{\mu}(1 + \mu)^{-1}J = J[\dot{\mu}(1 + \mu)^{-1}, J].
\]

It follows that the space of complex structures of \( E \) which are commensurable with \( J_0 \), when equipped with the natural almost complex structure \( \mathbb{J} \) defined by \( \mathbb{J}(\dot{J}) = J\dot{J} \), is naturally identified with an open subset of the vector space of \( J_0 \)-antilinear endomorphisms of \( E \), viewed as a vector space by setting \( i\mu = J_0 \circ \mu = -\mu \circ J_0 \).

### B.2. Symplectic complex structures

Let \( \omega_0 \) be a symplectic form on \( E \), i.e. an element of \( \Lambda^2 E^\ast \) such that \( \omega_0^m \neq 0 \). A complex structure \( J \) on \( E \) is said to be \( \omega_0 \)-compatible if \( \omega_0(J\cdot, J\cdot) = \omega_0 \) — so that \( g := \omega_0(\cdot, J\cdot) \) is symmetric — and if \( g_0 \) is positive definite. The set of \( \omega_0 \)-compatible complex structures of \( E \) will be denoted by \( \mathcal{C}(\omega_0) \).

We now fix \( J_0 \) in \( \mathcal{C}(\omega_0) \) and we denote by \( g_0 \) the (positive definite) inner product \( \omega_0(\cdot, J_0\cdot) \).

**Proposition B.2.1.** (i) Any \( J \) in \( \mathcal{C}(\omega_0) \) is commensurable to \( J_0 \).

(ii) A complex structure of \( E \) commensurable to \( J_0 \) belongs to \( \mathcal{C}(\omega_0) \) if and only if its Cayley transform \( \mu \) is symmetric and \( 1 \pm \mu \) are positive definite with respect to \( g_0 \).

**Proof.** (i) By definition, \( g := \omega_0(\cdot, J\cdot) \) is positive definite: we then have \( g = g_0(\cdot, S\cdot) \), where \( S \) is symmetric and positive definite with respect to \( g_0 \) and \( g \), and \( J = J_0S \). We thus get \( J + J_0 = J_0(1 + S) \), which is evidently invertible. By Proposition B.1.1, \( J \) is then commensurable to \( J_0 \).
The conclusion follows easily. □

We denote by $\text{Sp}(\omega_0)$ the symplectic group relative to $\omega_0$, defined as the group of linear transformations $\gamma$ of $E$ which preserve $\omega_0$, i.e. such that $\omega_0(\gamma \cdot , \cdot ) = \omega_0$. By considering $J_0$ and the corresponding inner product $g_0$ as in the previous section, the elements of $\text{Sp}(\omega_0)$ are also characterized by $\gamma^\ast J_0 \gamma = J_0$, where $\gamma^\ast$ stands for the adjoint of $\gamma$ with respect to $g_0$.

The Lie algebra, $\mathfrak{sp}(\omega_0)$, of $\text{Sp}(\omega_0)$ is the Lie algebra of linear transformations $A$ of $E$ such that $\omega_0(a \cdot , \cdot ) + \omega_0(\cdot , a \cdot ) = 0$. Equivalently, $a^\ast J_0 + J_0 a = 0$. This means that a linear transformation $a$ of $E$ belongs to $\mathfrak{sp}(\omega_0)$ if and only if $J_0 a$ is symmetric with respect to $g_0$ (in particular, the dimension of $\mathfrak{sp}(\omega_0)$, hence of $\text{Sp}(\omega_0)$, is equal to $m(2m + 1)$). It follows that $\text{Sp}(\omega_0)$ splits into the direct sum of the following two subspaces: (i) the space, $\mathfrak{u}(J_0)$, of those $a$ in $\text{End}(E)$ which are $g_0$-anti-symmetric and commute to $J_0$, hence the Lie algebra of the unitary group $U(J_0)$, which is the commutator of $J_0$ in $\text{Sp}(\omega_0)$; (ii) the space, $\mathfrak{m}(J_0)$, of those $a$ in $\text{End}(E)$ which are symmetric and anti-commute to $J_0$. The pair $(\mathfrak{u}(J_0), \mathfrak{m}(J_0))$ is a symmetric pair in the Lie algebra $\mathfrak{sp}(\omega_0)$, in the sense that

\begin{equation}
\mathfrak{u}(J_0), \mathfrak{m}(J_0) \subset \mathfrak{m}(J_0), \quad [\mathfrak{m}(J_0), \mathfrak{m}(J_0)] \subset \mathfrak{u}(J_0).
\end{equation}

The image of $\mathfrak{m}(J_0)$ by the exponential map from $\mathfrak{sp}(\omega_0)$ to $\text{Sp}(\omega_0)$ is a closed submanifold of $\text{Sp}(\omega_0)$ — namely the space of those $\gamma$ in $\text{Sp}(\omega_0)$ which are symmetric and satisfy $J_0 \gamma = \gamma^{-1} J_0$ — and the map $a \rightarrow \exp a$ is a diffeomorphism from $\mathfrak{m}(J_0)$ to its image. The product in $\text{Sp}(\omega_0)$ then induces a diffeomorphism from $U(J_0) \times \exp \mathfrak{m}(J_0)$ to $\text{Sp}(\omega_0)$, which makes the latter diffeomorphic to $U(J_0) \times \mathfrak{m}(J_0)$. In particular, $U(J_0)$ is a maximal (connected) compact subgp of $\text{Sp}(\omega_0)$.

The quotient $\text{Sp}(\omega_0)/U(J_0)$ is a hermitian symmetric spaces of non-compact type, hence a Hadamard space. At the origin $x_0$, the tangent space of $\text{Sp}(\omega_0)/U(J_0)$ is naturally identified with $\mathfrak{m}(J_0)$ and each point $x$ of $\text{Sp}(\omega_0)/U(J_0)$ is therefore of the form $x = \exp(a) \cdot x_0$, for a unique $a$ in $\mathfrak{m}(J_0)$. The curve $\exp(a) \cdot x_0$ is the unique geodesic in $\text{Sp}(\omega_0)/U(J_0)$ joining $J_0$ to $J$.

The group $\text{Sp}(\omega_0)$ acts on $\mathcal{C}(\omega_0)$, by

\begin{equation}
\gamma \cdot J = \gamma J \gamma^{-1}
\end{equation}

for any $\gamma$ in $\text{Sp}(\omega_0)$ and any $J$ in $\mathcal{C}(\omega_0)$. We then have

**Proposition B.2.2.** The action of $\text{Sp}(\omega_0)$ on $\mathcal{C}(\omega_0)$ given by (B.2.3) is transitive, hence identifies $\mathcal{C}(\omega_0)$ with the symmetric space $\text{Sp}(\omega_0)/U(J_0)$, for any chosen base-point $J_0$ in $\mathcal{C}(\omega_0)$.

**Proof.** The isotropy group of $J_0$ for the action (B.2.3) is clearly the unitary group $U(J_0)$. It then only remains to check that the action is transitive. Any element $J$ of $\mathcal{C}(\omega_0)$ satisfies $\omega_0(J \cdot , J \cdot ) = \omega_0$, hence can be viewed as an element of the group $\text{Sp}(\omega_0)$. The operator $S = J_0^{-1} J$ defined in the preceding section hence belongs to $\text{Sp}(\omega_0)$ and we clearly have that $SJ_0 = J_0 S^{-1}$. Since $S$ is positive definite with respect to $g$, it admits a (unique) positive
root, say $S^{1/2}$, and we again have that $S^{1/2}J_0 = J_0S^{-1/2}$; indeed, $J_0S^{-1/2}J_0$ is also a positive root of $S$, hence is equal to $S^{1/2}$. It follows that $S^{1/2}$ also belongs to $\text{Sp}(\omega_0)$ and $J = S^{-1/2}J_0S^{1/2}$. We conclude that $\mathcal{C}(\omega_0)$ coincides with the orbit of $J_0$ under the action of $\text{Sp}(\omega_0)$.

Via the above identification, the space $\mathcal{C}(\omega_0)$ admits a homogeneous Kähler structure $(g, J, \kappa)$ which is actually independent of the chosen base point $J_0$ and is described as follows. At any point $J$ of $\mathcal{C}(\omega_0)$, the tangent space $T_J\mathcal{C}(\omega_0)$ is naturally identified with the space of endomorphisms $A$ of $E$ which anticommute with $J$ and are symmetric with respect to the metric $g_{\omega_0}(\cdot, J\cdot)$ and we then have

\begin{equation}
J A = JA, \quad g(A_1, A_2) = \frac{1}{2} \text{tr}(A_1A_2), \quad \kappa(A_1, A_2) = \frac{1}{2} \text{tr}(JA_1A_2).
\end{equation}

Remark B.2.1. Any $J$ in $\mathcal{C}(\omega_0)$ is related to the chosen base-point $J_0$ by $J = \exp(a)J_0\exp(-a)$ for a uniquely defined element $a$ in $T_J\mathcal{C}(\omega_0)$. We then have that $S = J_0^{-1}J = \exp(-2a)$ and, by (B.2.1), $a$ is related to the Cayley transform $\mu$ of $J$ with respect to $J_0$ by $\mu = \tanh a$. 


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